Lecture Course
GROUP THEORY AND TOPOLOGY

Donostia - San Sebastian

23-26 August 2018

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Basque Foundation for Science

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SPACE-GROUP SYMMETRY

SYMMETRY DATABASES OF BILBAO CRYSTALLOGRAPHIC SERVER

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SPACE GROUPS

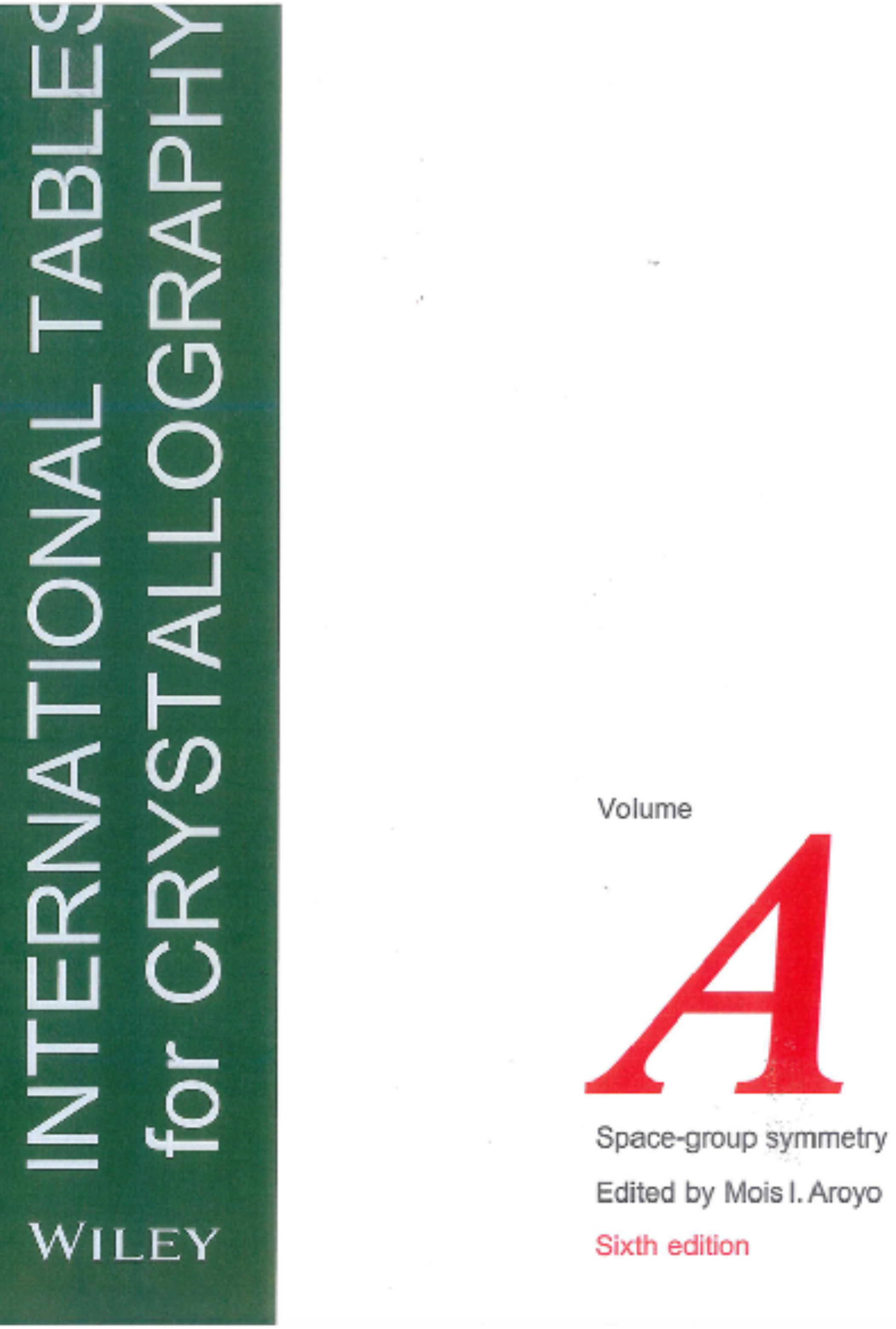
Crystal pattern: infinite, idealized crystal structure (without disorder, dislocations, impurities, etc.)

Space group G: The set of all symmetry operations (isometries) of a crystal pattern

Translation subgroup $H \lhd G$: The infinite set of all translations that are symmetry operations of the crystal pattern

Point group of the space groups P_G: The factor group of the space group G with respect to the translation subgroup T: $P_G \cong G/H$

INTERNATIONAL TABLES FOR CRYSTALLOGRAPHY
VOLUME A: SPACE-GROUP SYMMETRY



Extensive tabulations and illustrations of the 17 plane groups and of the 230 space groups

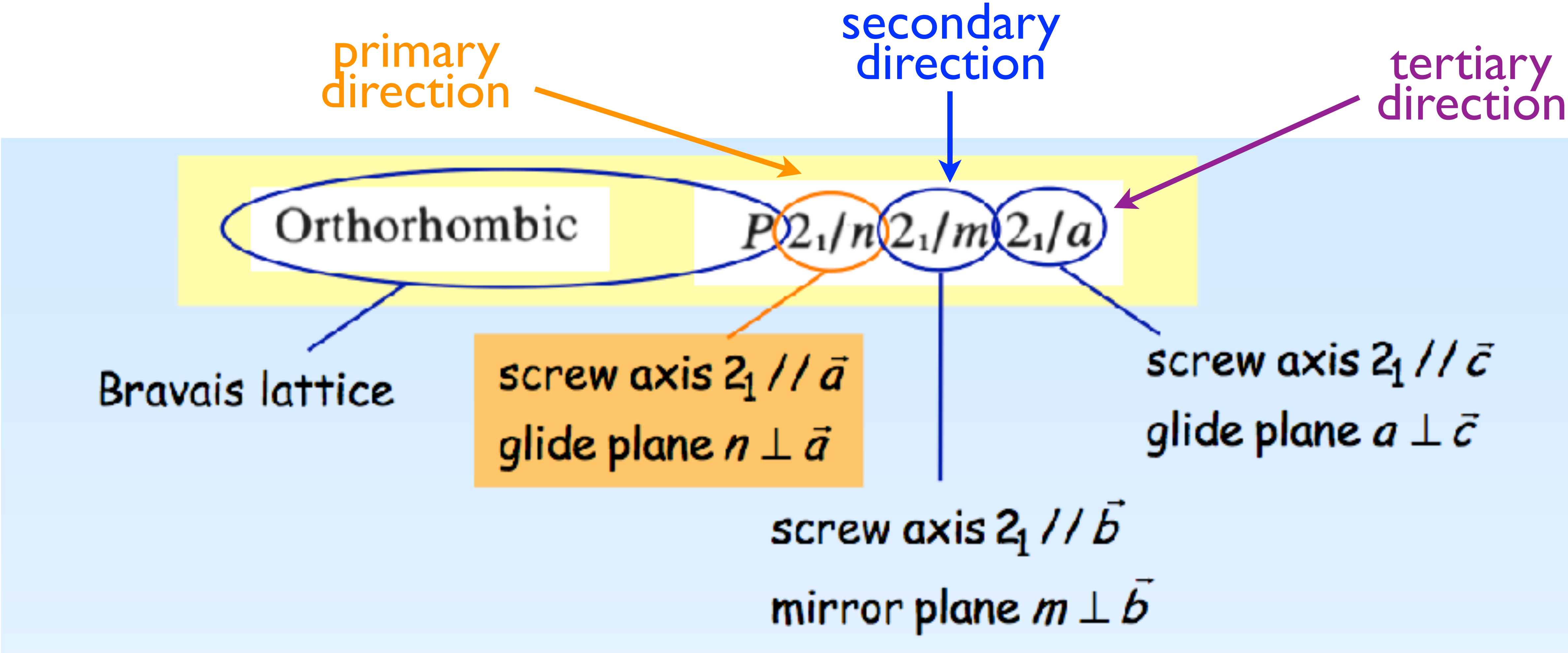
- headline with the relevant group symbols;
- diagrams of the symmetry elements and of the general position;
- specification of the origin and the asymmetric unit;
- list of symmetry operations;
- generators;
- general and special positions with multiplicities, site symmetries, coordinates and reflection conditions;
- symmetries of special projections;

HERMANN-MAUGUIN SYMBOLISM

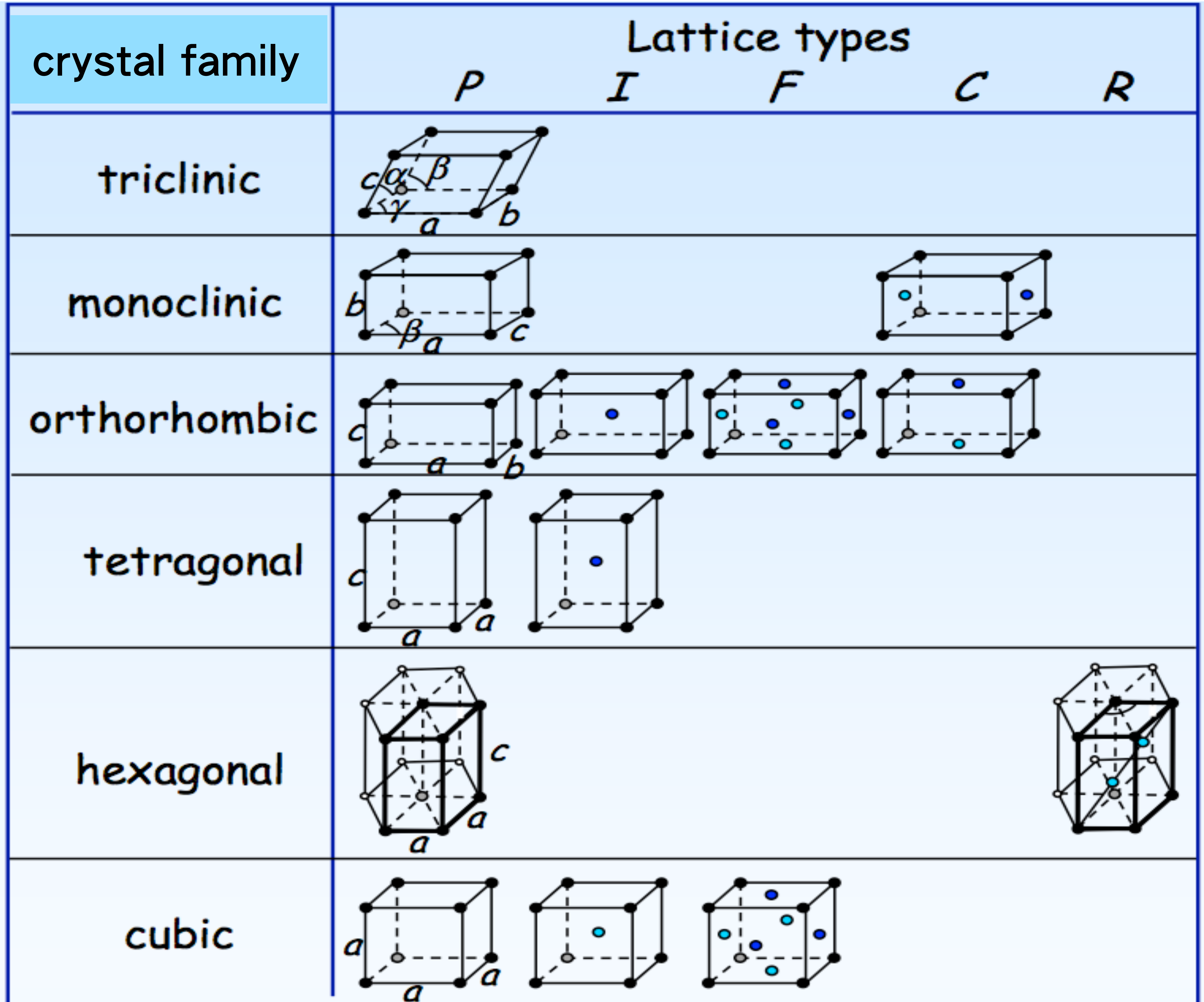
Hermann-Mauguin symbols for space groups

- centring type
- symmetry elements along *primary*, *secondary* and *ternary* symmetry directions
  - rotations: by the axes of rotation
  - planes: by the normals to the planes
- rotations/planes along the same direction

A direction is called a ***symmetry direction*** of a crystal structure if it is parallel to an axis of rotation or rotoinversion or if it is parallel to the normal of a reflection plane.



14 Bravais Lattices



Hermann-Mauguin symbols for space groups

Directions that belong to the same set of equivalent symmetry directions are collected between braces. The first entry in each set is taken as the representative of that set.

Lattice	Symmetry direction (position in Hermann–Mauguin symbol)		
	Primary	Secondary	Tertiary
Triclinic	None		
Monoclinic*	[010] ('unique axis b') [001] ('unique axis c')		
Orthorhombic	[100]	[010]	[001]
Tetragonal	[001]	$\left\{\begin{matrix}[100]\\ [010]\end{matrix}\right\}$	$\left\{\begin{matrix}[1\bar{1}0]\\ [110]\end{matrix}\right\}$
Hexagonal	[001]	$\left\{\begin{matrix}[100]\\ [010]\\ [\bar{1}\bar{1}0]\end{matrix}\right\}$	$\left\{\begin{matrix}[1\bar{1}0]\\ [120]\\ [\bar{2}\bar{1}0]\end{matrix}\right\}$
Rhombohedral (hexagonal axes)	[001]	$\left\{\begin{matrix}[100]\\ [010]\\ [\bar{1}\bar{1}0]\end{matrix}\right\}$	
Rhombohedral (rhombohedral axes)	[111]	$\left\{\begin{matrix}[1\bar{1}0]\\ [01\bar{1}]\\ [\bar{1}01]\end{matrix}\right\}$	
Cubic	$\left\{\begin{matrix}[100]\\ [010]\\ [001]\end{matrix}\right\}$	$\left\{\begin{matrix}[111]\\ [1\bar{1}\bar{1}]\\ [\bar{1}1\bar{1}]\\ [\bar{1}\bar{1}1]\end{matrix}\right\}$	$\left\{\begin{matrix}[1\bar{1}0] & [110]\\ [01\bar{1}] & [011]\\ [\bar{1}01] & [101]\end{matrix}\right\}$

SPACE-GROUP SYMMETRY OPERATIONS

Symmetry Operations Characteristics

TYPE of the symmetry operation

preserve or not ***handedness***

SCREW/GLIDE component

GEOMETRIC ELEMENT

ORIENTATION of the geometric element

LOCATION of the geometric element

Crystallographic symmetry operations

characteristics:

fixed points of isometries $(W,w)X_f=X_f$

geometric elements

Types of isometries preserve handedness

identity: the whole space fixed

translation t: no fixed point $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{t}$

rotation: one line fixed

rotation axis

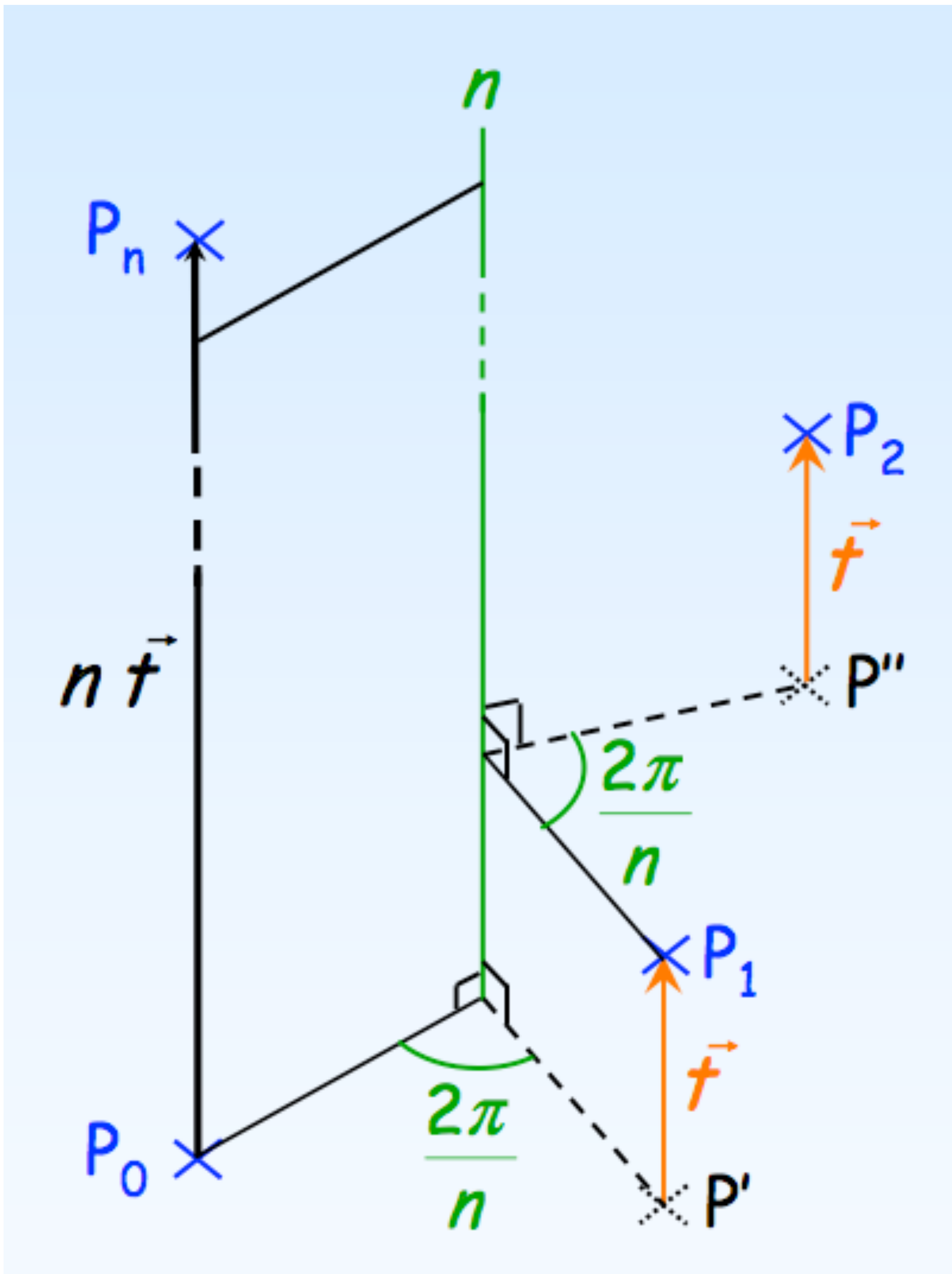
$$\phi = k \times 360^\circ / N$$

screw rotation: no fixed point

screw axis

screw vector

Screw rotation



n-fold rotation followed by a fractional translation $\frac{p}{n}\mathbf{t}$ parallel to the rotation axis

Its application *n* times results in a translation parallel to the rotation axis

Types of isometries do not preserve handedness

characteristics: fixed points of isometries $(W,w)X_f=X_f$
geometric elements

roto-inversion: centre of roto-inversion fixed
roto-inversion axis

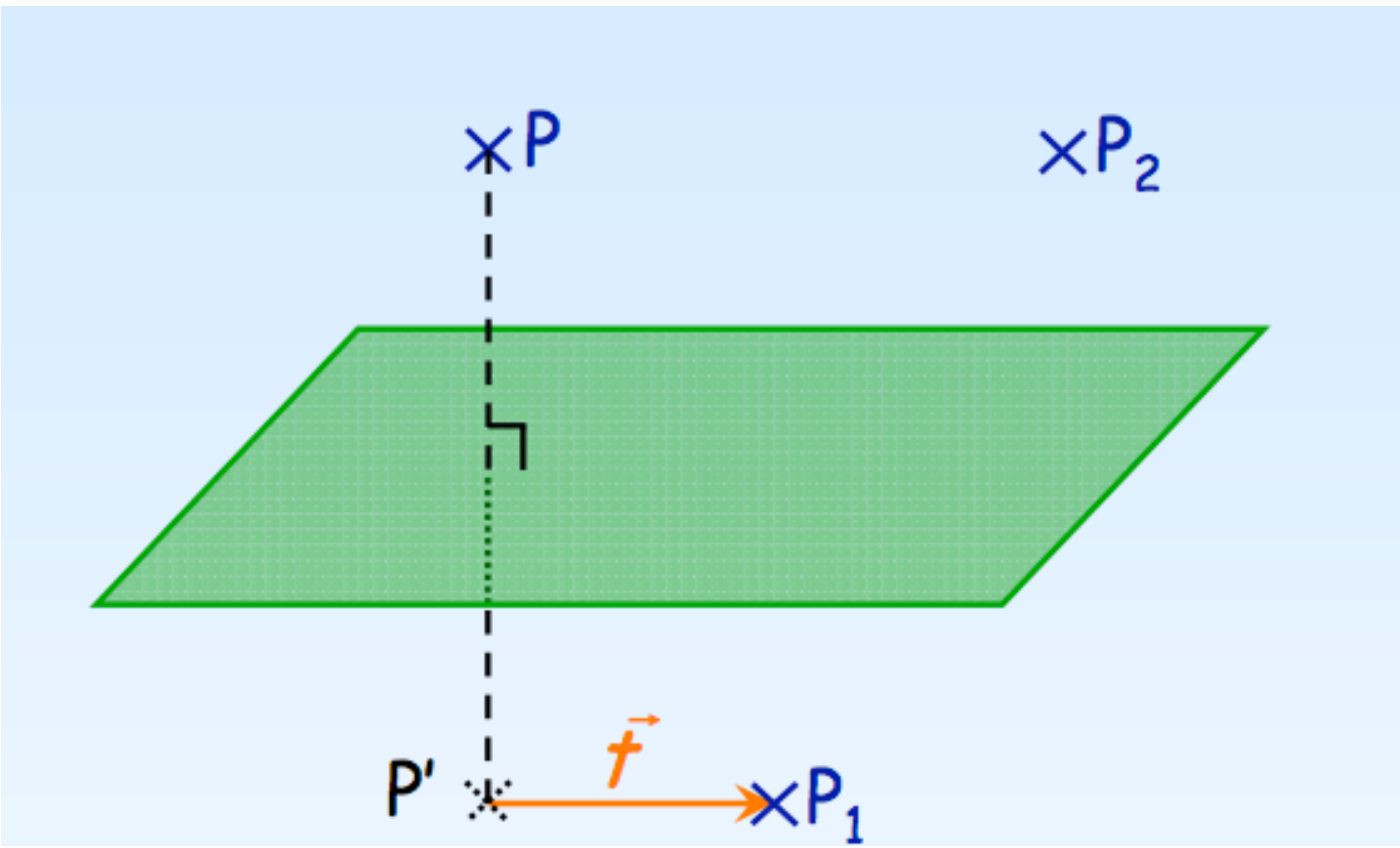
inversion: centre of inversion fixed

reflection: plane fixed
reflection/mirror plane

glide reflection: no fixed point
glide plane glide vector

Crystallographic symmetry operations

Glide plane



reflection followed by a fractional translation $\frac{1}{2}\mathbf{t}$ parallel to the plane

Its application 2 times results in a translation parallel to the plane

Description of isometries

coordinate system: $\{O, \mathbf{a}, \mathbf{b}, \mathbf{c}\}$

isometry:

$\tilde{x} = F_1(x,y,z)$

$$\left|\begin{array}{lll} \tilde{x} & = & W_{11}\,x + W_{12}\,y + W_{13}\,z + w_1 \\ \tilde{y} & = & W_{21}\,x + W_{22}\,y + W_{23}\,z + w_2 \\ \tilde{z} & = & W_{31}\,x + W_{32}\,y + W_{33}\,z + w_3 \end{array}\right.$$

Matrix-column presentation of isometries

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} = \begin{pmatrix}W_{11} W_{12} W_{13}\\ W_{21} W_{22} W_{23}\\ W_{31} W_{32} W_{33}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix}$$

linear/matrix part

translation column part

$$\tilde{\boldsymbol{x}} = \boldsymbol{W}\,\boldsymbol{x} + \boldsymbol{w}$$

$$\tilde{\boldsymbol{x}} = (\,\boldsymbol{W},\,\boldsymbol{w}\,)\,\boldsymbol{x} \quad \text{or} \quad \tilde{\boldsymbol{x}} = \{\,\boldsymbol{W}\,|\,\boldsymbol{w}\,\}\,\boldsymbol{x}$$

matrix-column pair

Seitz symbol

EXERCISES

Problem 2.1

Referred to an 'orthorhombic' coordinated system ($a \neq b \neq c$; $\alpha=\beta=\gamma=90$) two symmetry operations are represented by the following matrix-column pairs:

$$(W_1,w_1)=\left(\begin{array}{ccc|c} -1 & & & 0 \\ & 1 & & 0 \\ & & -1 & 0 \end{array}\right) \qquad (W_2,w_2)=\left(\begin{array}{ccc|c} -1 & & & 1/2 \\ & 1 & & 0 \\ & & -1 & 1/2 \end{array}\right)$$

Determine the images X_i of a point X under the symmetry operations (W_i,w_i) where

$$X=\begin{pmatrix} 0,70 \\ 0,31 \\ 0,95 \end{pmatrix}$$

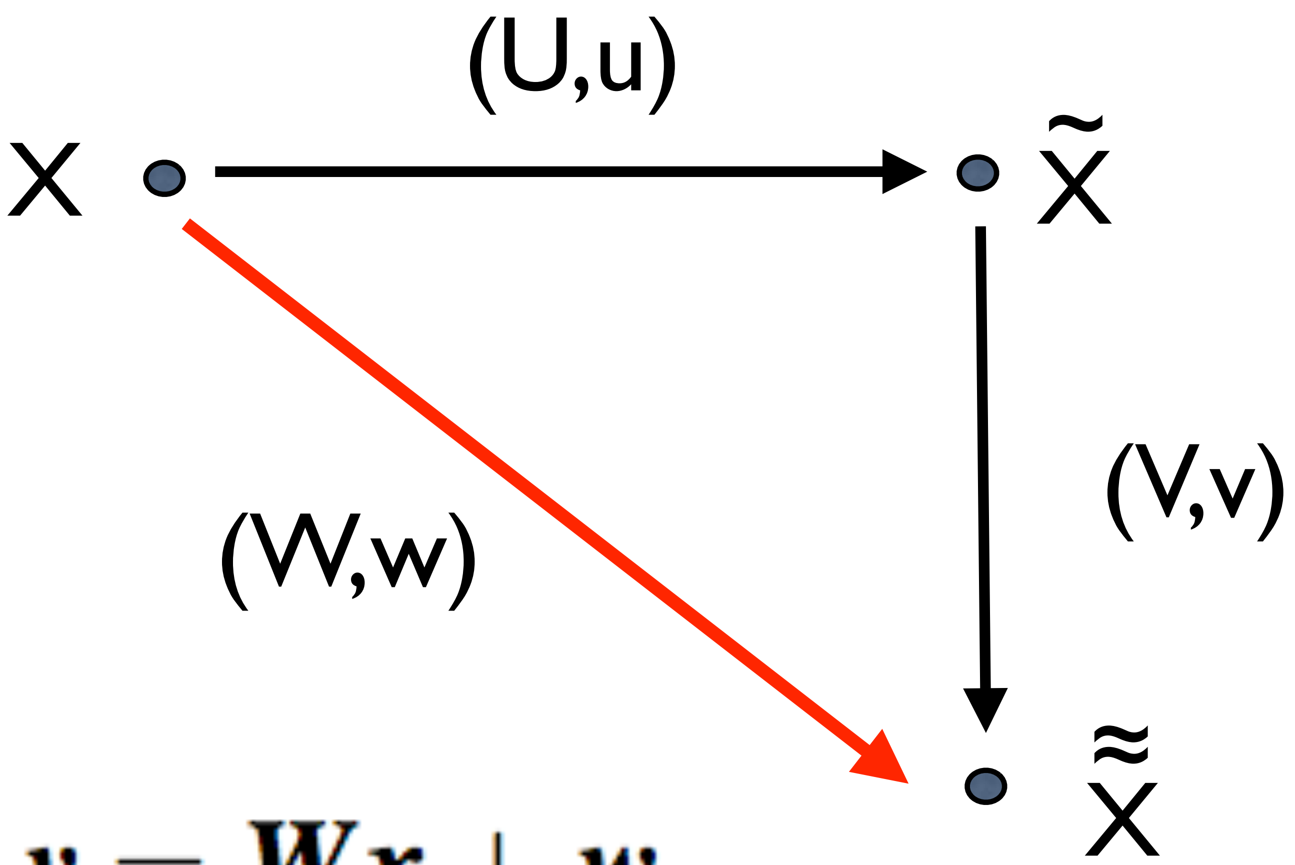
Can you guess what is the geometric 'nature' of (W_1,w_1)?

And of (W_2,w_2)?

Hint:

A drawing could be rather helpful

Combination of isometries



$$\tilde{\boldsymbol{x}} = \boldsymbol{U}\boldsymbol{x} + \boldsymbol{u};$$

$$\tilde{\tilde{\boldsymbol{x}}} = \boldsymbol{V}\tilde{\boldsymbol{x}} + \boldsymbol{v};$$

$$\tilde{\tilde{\boldsymbol{x}}} = \boldsymbol{V}(\boldsymbol{U}\boldsymbol{x} + \boldsymbol{u}) + \boldsymbol{v};$$

$$\tilde{\tilde{\boldsymbol{x}}} = \boldsymbol{V}\boldsymbol{U}\boldsymbol{x} + \boldsymbol{V}\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{W}\boldsymbol{x} + \boldsymbol{w}.$$

$$\tilde{\tilde{\boldsymbol{x}}} = (\boldsymbol{V},\ \boldsymbol{v})\,\tilde{\boldsymbol{x}} = (\boldsymbol{V},\ \boldsymbol{v})(\boldsymbol{U},\ \boldsymbol{u})\boldsymbol{x} = (\boldsymbol{W},\ \boldsymbol{w})\boldsymbol{x}.$$

$$(\boldsymbol{W},\ \boldsymbol{w}) = (\boldsymbol{V},\ \boldsymbol{v})(\boldsymbol{U},\ \boldsymbol{u}) = (\boldsymbol{V}\boldsymbol{U},\ \boldsymbol{V}\,\boldsymbol{u} + \boldsymbol{v}).$$

Problem 2.1(cont)

Consider the matrix-column pairs of the two symmetry operations:

$$(W_1,w_1)=\left(\begin{array}{ccc|c} 0 & -1 & & 0 \\ 1 & 0 & & 0 \\ & & 1 & 0 \end{array}\right) \qquad (W_2,w_2)=\left(\begin{array}{ccc|c} -1 & & & 1/2 \\ & 1 & & 0 \\ & & -1 & 1/2 \end{array}\right)$$

Determine and compare the matrix-column pairs of the combined symmetry operations:

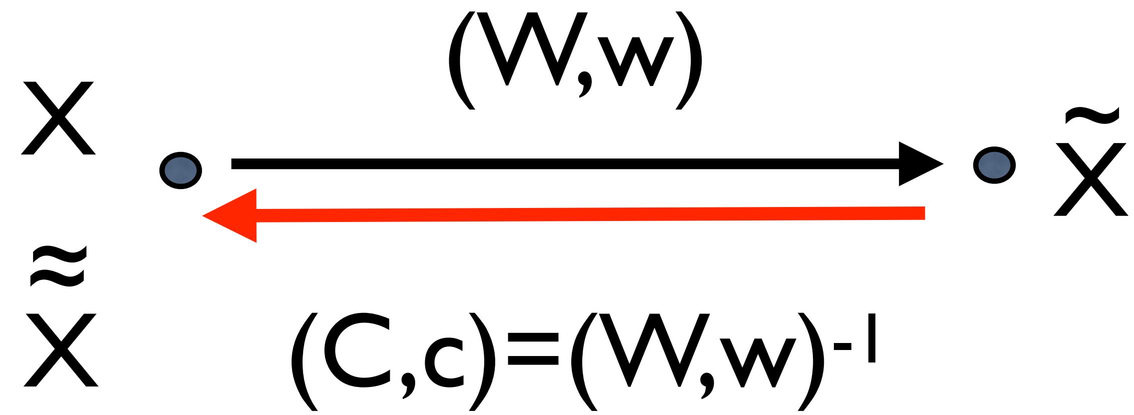
$$(W,w)=(W_1,w_1)(W_2,w_2)$$

$$(W,w)'=(W_2,w_2)(W_1,w_1)$$

combination of isometries:

$$(\boldsymbol{W}_2,\ \boldsymbol{w}_2)(\boldsymbol{W}_1,\ \boldsymbol{w}_1)=(\boldsymbol{W}_2\boldsymbol{W}_1,\ \boldsymbol{W}_2\boldsymbol{w}_1+\boldsymbol{w}_2)$$

Inverse isometries



$\boldsymbol{I}$ = 3x3 identity matrix

$\boldsymbol{o}$ = zero translation column

$$(C,c)(W,w) = (\boldsymbol{I},\boldsymbol{o})$$

$$(C,c)(W,w) = (CW, Cw+c)$$

$CW=\boldsymbol{I}$ → $C=W^{-1}$

$Cw+c=\boldsymbol{o}$ → $c=-Cw=-W^{-1}w$

Determine the inverse symmetry operations $(W_1,w_1)^{-1}$ and $(W_2,w_2)^{-1}$ where

$$(W_1,w_1)=\left(\begin{array}{ccc|c} 0 & -1 & & 0 \\ 1 & 0 & & 0 \\ & & 1 & 0 \end{array}\right) \qquad (W_2,w_2)=\left(\begin{array}{ccc|c} -1 & & & 1/2 \\ & 1 & & 0 \\ & & -1 & 1/2 \end{array}\right)$$

Determine the inverse symmetry operation $(W,w)^{-1}$

$$(W,w)=(W_1,w_1)(W_2,w_2)$$

inverse of isometries:

$$(\boldsymbol{W},\, \boldsymbol{w})^{-1} = (\boldsymbol{W}^{-1},\, -\boldsymbol{W}^{-1}\,\boldsymbol{w})$$

Short-hand notation for the description of isometries

isometry:

$$X \bullet \xrightarrow{(W,w)} \bullet \tilde{X}$$

$$\left| \begin{array}{lcl} \tilde{x} & = & W_{11}\,x + W_{12}\,y + W_{13}\,z + w_1 \\ \tilde{y} & = & W_{21}\,x + W_{22}\,y + W_{23}\,z + w_2 \\ \tilde{z} & = & W_{31}\,x + W_{32}\,y + W_{33}\,z + w_3 \end{array} \right.$$

notation rules:

-left-hand side: omitted
-coefficients 0, +1, -1
-different rows in one line

examples:

-1			1/2
	1		0
		-1	1/2

$\longrightarrow$ { -x+1/2, y, -z+1/2
$\bar{x}$+1/2, y, $\bar{z}$+1/2

EXERCISES

Problem 2.2

Construct the matrix-column pair (W,w) of the following coordinate triplets:

(1) x,y,z (2) -x,y+1/2,-z+1/2

(3) -x,-y,-z (4) x,-y+1/2, z+1/2

Matrix formalism: 4x4 matrices

$$\boldsymbol{x} \rightarrow \mathbb{x} = \left(\begin{array}{c} x \\ y \\ z \\ \hline 1 \end{array}\right) \; ; \; \tilde{\boldsymbol{x}} \rightarrow \tilde{\mathbb{x}} = \left(\begin{array}{c} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \hline 1 \end{array}\right)$$

augmented matrices:

$$(\boldsymbol{W}, \boldsymbol{w}) \rightarrow \mathbb{W} = \left(\begin{array}{ccc|c} & \boldsymbol{W} & & \boldsymbol{w} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

point $X \longrightarrow$ point $\tilde{X}$:

$$\tilde{\mathbb{x}} = \mathbb{W}\,\mathbb{x} \qquad \left(\begin{array}{c} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \hline 1 \end{array}\right) = \left(\begin{array}{ccc|c} & \boldsymbol{W} & & \boldsymbol{w} \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c} x \\ y \\ z \\ \hline 1 \end{array}\right)$$

4x4 matrices: general formulae

point $X \longrightarrow$ point $\tilde{X}$:

$$\tilde{\mathbb{x}} = \mathbb{W}\,\mathbb{x} \qquad \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & \boldsymbol{W} & & \boldsymbol{w} \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x \\ y \\ z \\ \hline 1 \end{pmatrix}$$

combination and inverse of isometries:

$$(\mathbb{W})^{-1} = (\mathbb{W}^{-1}) \qquad \mathbb{W}^{-1} = \left(\begin{array}{ccc|c} & \boldsymbol{W}^{-1} & & -\boldsymbol{W}^{-1}\boldsymbol{w} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbb{W}_3 = \mathbb{W}_2\,\mathbb{W}_1$$

PRESENTATION OF SPACE-GROUP SYMMETRY OPERATIONS

IN INTERNATIONAL TABLES FOR CRYSTALLOGRAPHY, VOL. A

Space group *Cmm*2 (No. 35)

How are the symmetry operations represented in ITA ?

Diagram of symmetry elements

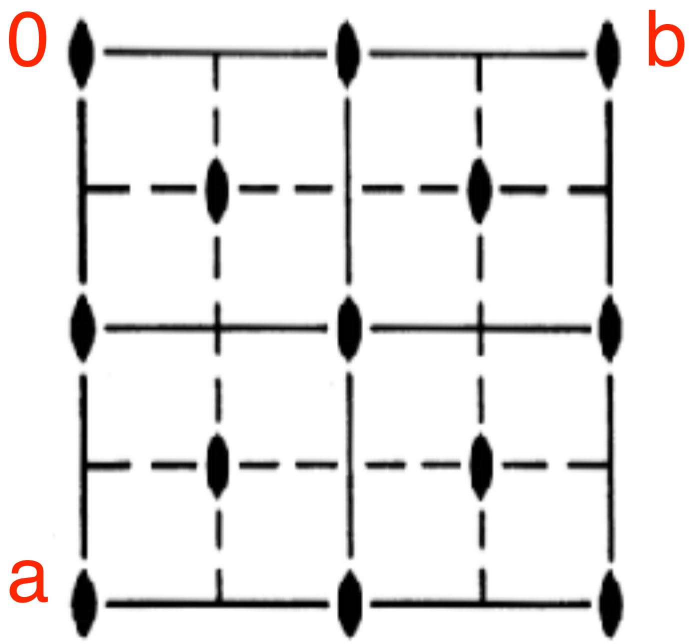
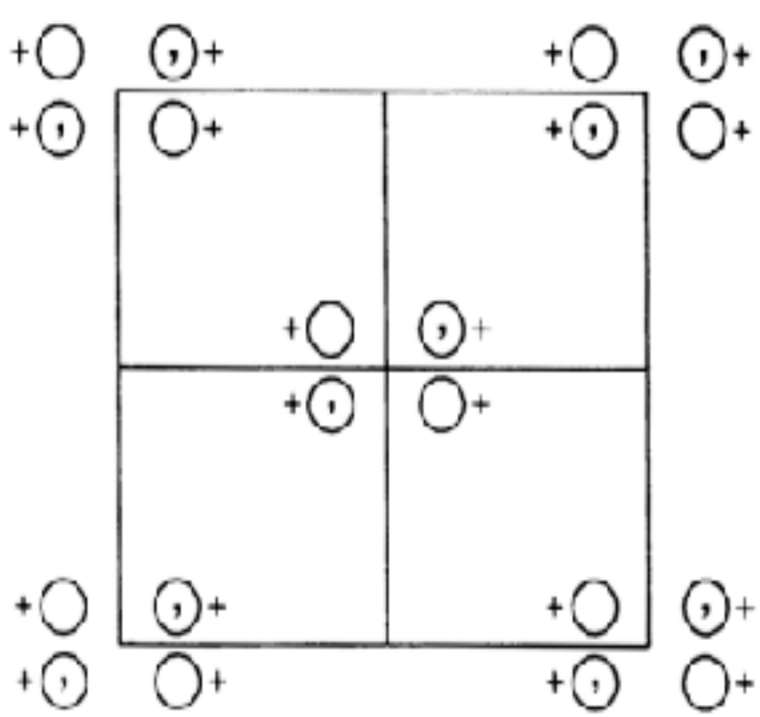


Diagram of general position points



Symmetry operations

For (0,0,0)+ set

(1) 1	(2) 2 $0,0,z$	(3) m $x,0,z$	(4) m $0,y,z$

For $(\frac{1}{2},\frac{1}{2},0)+$ set

(1) $t(\frac{1}{2},\frac{1}{2},0)$	(2) 2 $\frac{1}{4},\frac{1}{4},z$	(3) a $x,\frac{1}{4},z$	(4) b $\frac{1}{4},y,z$

General Position

Coordinates

$(0,0,0)+ \quad (\frac{1}{2},\frac{1}{2},0)+$

8	f	1	(1) x,y,z	(2) $\bar{x},\bar{y},z$	(3) $x,\bar{y},z$	(4) $\bar{x},y,z$

General position

(i) coordinate triplets of an image point $\tilde{X}$ of the original point $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under (W,w) of G

-presentation of infinite image points $\tilde{X}$ under the action of (W,w) of G

(ii) short-hand notation of the matrix-column pairs (W,w) of the symmetry operations of G

-presentation of infinite symmetry operations of G

$(W,w) = (I,t_n)(W,w_0),\ 0 \leq w_{i0} < 1$

General Position of Space groups (infinite order)

Coset decomposition $G:T_G$

General position

$(I,0)$	(W_2,w_2)	...	(W_m,w_m)	...	(W_i,w_i)
(I,t_1)	(W_2,w_2+t_1)	...	(W_m,w_m+t_1)	...	(W_i,w_i+t_1)
(I,t_2)	(W_2,w_2+t_2)	...	(W_m,w_m+t_2)	...	(W_i,w_i+t_2)
...	...	...	...	...	...
(I,t_j)	(W_2,w_2+t_j)	...	(W_m,w_m+t_j)	...	(W_i,w_i+t_j)
...	...	...	...	...	...

Symmetry operations expressed in x,y,z notation

Factor group G/T_G

isomorphic to the point group P_G of G

Point group $P_G = \{I, W_2, W_3, \ldots, W_i\}$

Symmetry Operations Block

TYPE of the symmetry operation

SCREW/GLIDE component

ORIENTATION of the geometric element

LOCATION of the geometric element

GEOMETRIC INTERPRETATION OF THE MATRIX-COLUMN PRESENTATION OF THE SYMMETRY OPERATIONS

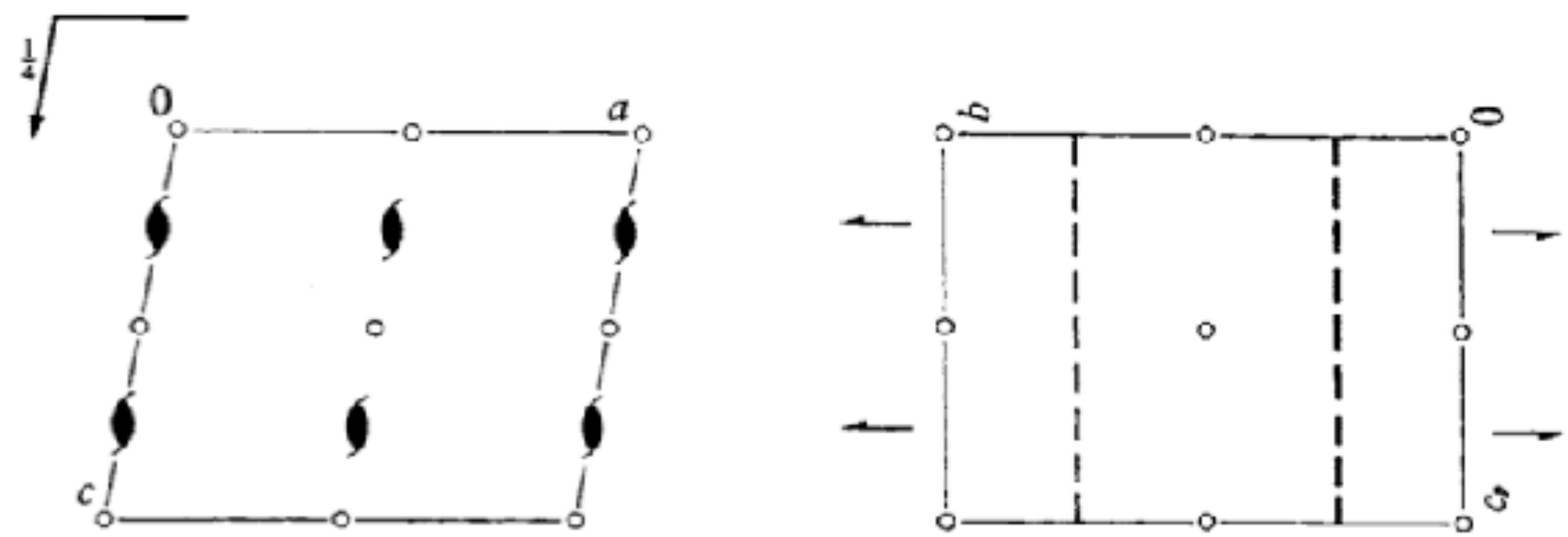
Space group $P2_1/c$ (No. 14)

$P2_1/c$ $\quad C_{2h}^5$ $\quad 2/m$ $\quad$ Monoclinic

No. 14 $\quad P12_1/c1$ $\quad$ Patterson sy

UNIQUE AXIS b, CELL CHOICE 1

EXAMPLE



Generators selected (1); $t(1,0,0)$; $t(0,1,0)$; $t(0,0,1)$; (2); (3)

Positions

Multiplicity,
Wyckoff letter,
Site symmetry

Coordinates

Matrix-column presentation

4	e	1	(1) x,y,z	(2) $\bar{x},y+\frac{1}{2},\bar{z}+\frac{1}{2}$	(3) $\bar{x},\bar{y},\bar{z}$	(4) $x,\bar{y}+\frac{1}{2},z+\frac{1}{2}$

Geometric interpretation

Symmetry operations

(1) 1 $\quad$ (2) $2(0,\frac{1}{2},0)$ $\quad 0,y,\frac{1}{4}$ $\quad$ (3) $\bar{1}$ $\quad 0,0,0$ $\quad$ (4) c $\quad x,\frac{1}{4},z$

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Space-group symmetry

Magnetic Symmetry and Applications

Group-Subgroup Relations of Space Groups

Representations and Applications

Solid State Theory Applications

Structure Utilities

Subperiodic Groups: Layer, Rod and Frieze Groups

Structure Databases

Raman and Hyper-Raman scattering

Point-group symmetry

Plane-group symmetry

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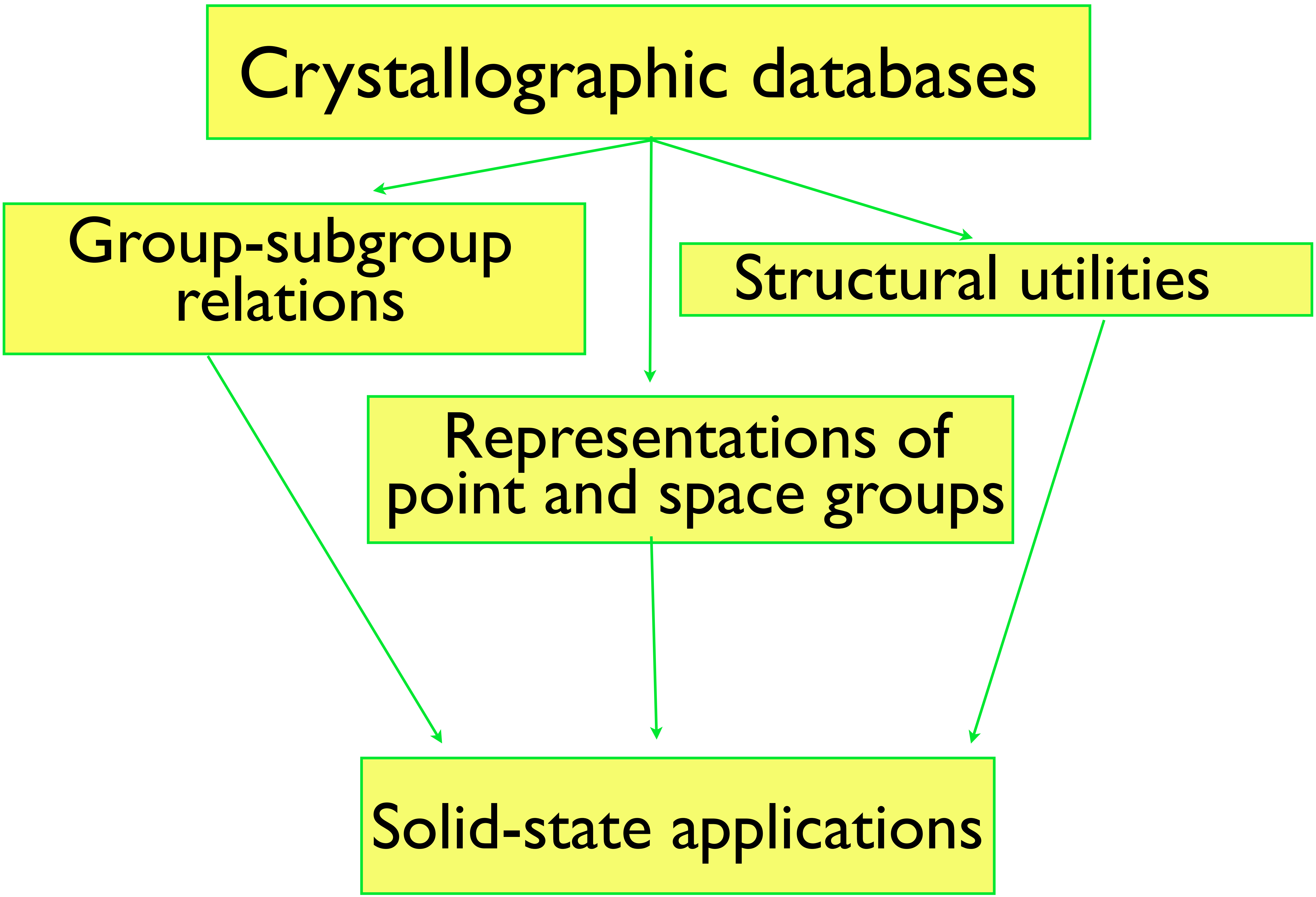
20-21 August 2018

ews:

- **New Article in Nature**
  07/2017: Bradlyn *et al.* "Topological quantum chemistry" *Nature* (2017). **547**, 298-305.
- **New program: BANDREP**
  04/2017: Band representations and Elementary Band representations of Double Space Groups.
- **New section: Double point and space groups**
  - **New program: DGENPOS**
    04/2017: General positions of Double Space Groups
  - **New program: REPRESENTATIONS DPG**
    04/2017: Irreducible representations of

Crystallographic Databases

International Tables for Crystallography


Crystallographic databases
Group-subgroup relations
Structural utilities
Representations of point and space groups
Solid-state applications

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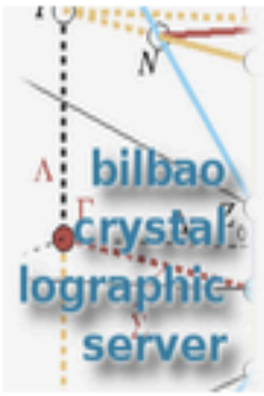

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  - New program: DGENPOS
    04/2017: General positions of Double Space Groups
  - New program: REPRESENTATIONS DPG
    04/2017: Irreducible representations of

Space-group symmetry

GENPOS	Generators and General Positions of Space Groups
WYCKPOS	Wyckoff Positions of Space Groups
HKLCOND	Reflection conditions of Space Groups
MAXSUB	Maximal Subgroups of Space Groups
SERIES	Series of Maximal Isomorphic Subgroups of Space Groups
WYCKSETS	Equivalent Sets of Wyckoff Positions
NORMALIZER	Normalizers of Space Groups
KVEC	The k-vector types and Brillouin zones of Space Groups
SYMMETRY OPERATIONS	Geometric interpretation of matrix column representations of symmetry operations
IDENTIFY GROUP	Identification of a Space Group from a set of generators in an arbitrary setting

Structure Utilities

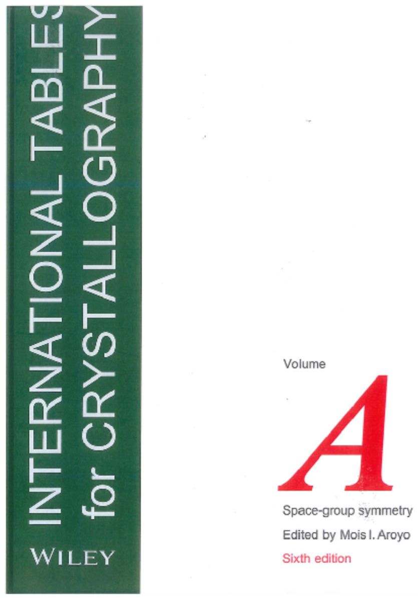
Subperiodic Groups: Layer, Rod and Frieze Groups

Structure Databases

Raman and Hyper-Raman scattering

Point-group symmetry

Plane-group symmetry

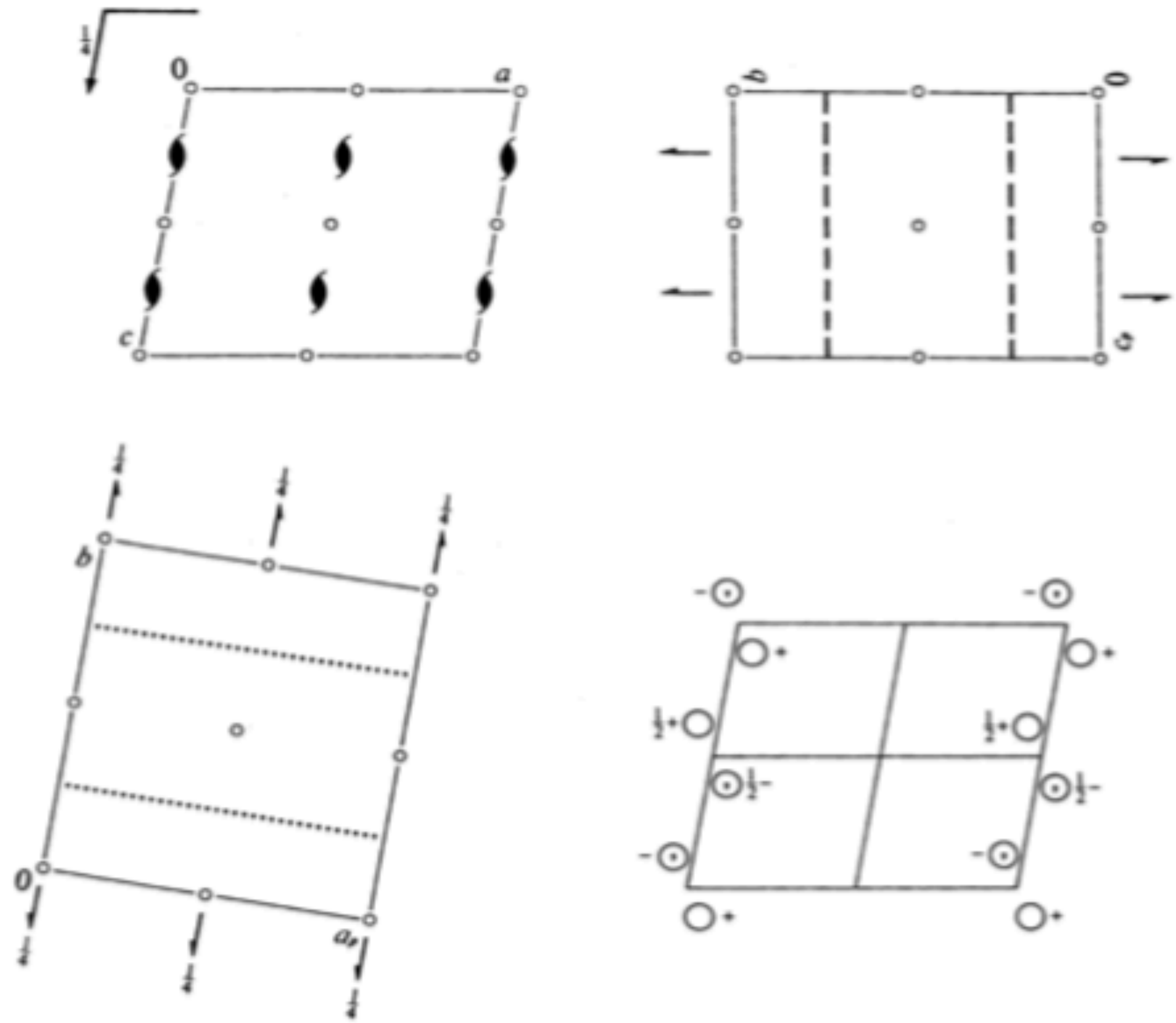


International Tables for Crystallography (2016). Vol. A, Space group 14, pp. 252–259.

$P2_1/c$ C_{2h}^5 $2/m$ Monoclinic

No. 14 $P\,1\,2_1/c\,1$ Patterson symmetry $P\,1\,2/m\,1$

UNIQUE AXIS b, CELL CHOICE 1



Origin at $\bar{1}$

Asymmetric unit $0 \le x \le 1;\quad 0 \le y \le \frac{1}{4};\quad 0 \le z \le 1$

Symmetry operations

(1) 1 (2) $2(0,\frac{1}{2},0)$ $0,y,\frac{1}{4}$ (3) $\bar{1}$ $0,0,0$ (4) c $x,\frac{1}{4},z$

Generators selected (1); $t(1,0,0)$; $t(0,1,0)$; $t(0,0,1)$; (2); (3)

Positions

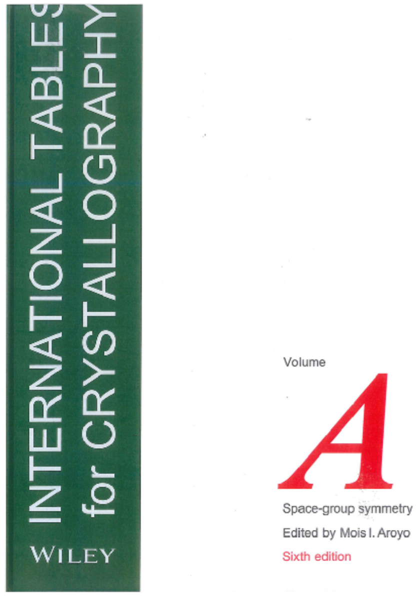
Multiplicity, Wyckoff letter, Site symmetry			Coordinates				Reflection conditions
							General:
4	e	1	(1) x,y,z	(2) $\bar{x},y+\frac{1}{2},\bar{z}+\frac{1}{2}$	(3) $\bar{x},\bar{y},\bar{z}$	(4) $x,\bar{y}+\frac{1}{2},z+\frac{1}{2}$	$h0l$: $l=2n$ $0k0$: $k=2n$ $00l$: $l=2n$
							Special: as above, plus
2	d	$\bar{1}$	$\frac{1}{2},0,\frac{1}{2}$	$\frac{1}{2},\frac{1}{2},0$			hkl: $k+l=2n$
2	c	$\bar{1}$	$0,0,\frac{1}{2}$	$0,\frac{1}{2},0$			hkl: $k+l=2n$
2	b	$\bar{1}$	$\frac{1}{2},0,0$	$\frac{1}{2},\frac{1}{2},\frac{1}{2}$			hkl: $k+l=2n$
2	a	$\bar{1}$	$0,0,0$	$0,\frac{1}{2},\frac{1}{2}$			hkl: $k+l=2n$

Symmetry of special projections

Along [001] $p2gm$
$\mathbf{a}'=\mathbf{a}_p$ $\mathbf{b}'=\mathbf{b}$
Origin at $0,0,z$

Along [100] $p2gg$
$\mathbf{a}'=\mathbf{b}$ $\mathbf{b}'=\mathbf{c}_p$
Origin at $x,0,0$

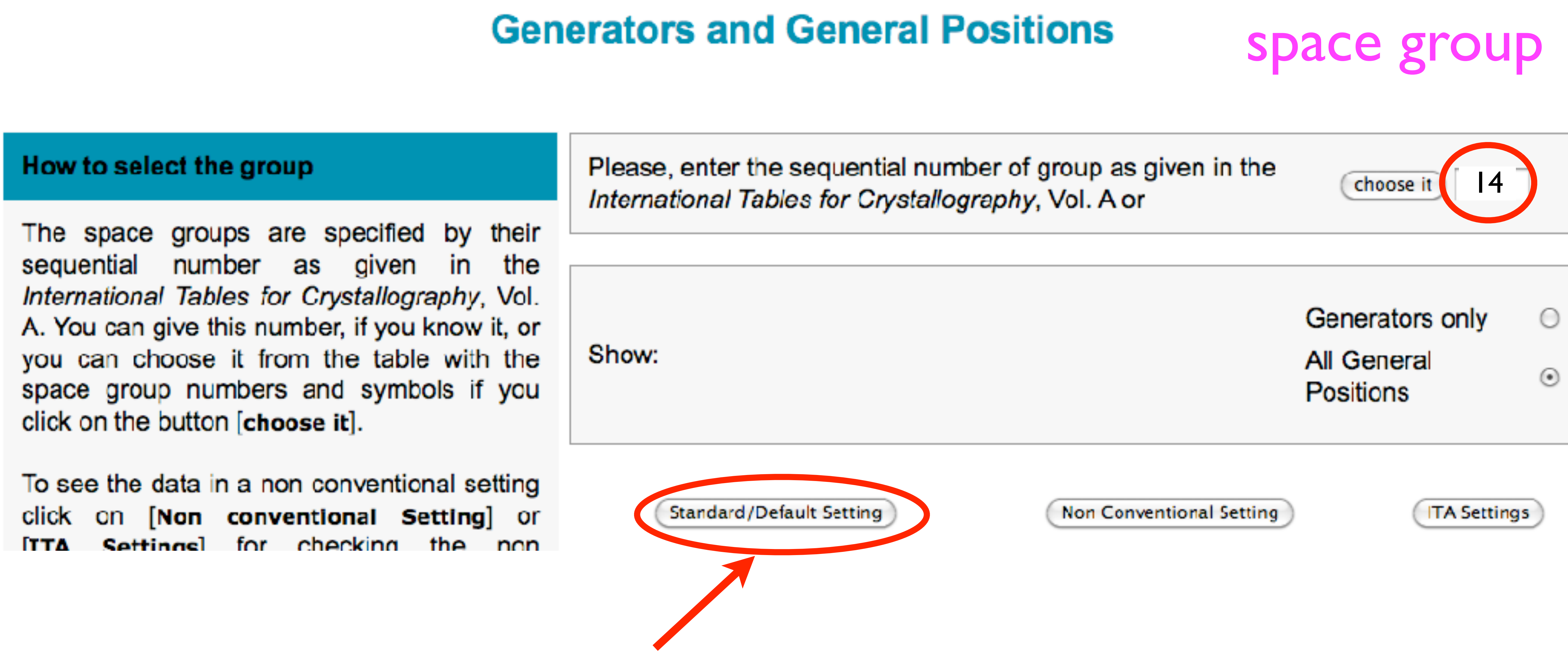
Along [010] $p2$
$\mathbf{a}'=\frac{1}{2}\mathbf{c}$ $\mathbf{b}'=\mathbf{a}$
Origin at $0,y,0$



Bilbao Crystallographic Server

Problem: Matrix-column presentation
Geometrical interpretation

GENPOS



Example GENPOS: Space group $P2_1/c$ (14)

Space-group symmetry operations

short-hand notation

matrix-column presentation $\begin{pmatrix} W_{11} W_{12} W_{13} \\ W_{21} W_{22} W_{23} \\ W_{31} W_{32} W_{33} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

Geometric interpretation

Seitz symbols

General Positions of the Group 14 ($P2_1/c$) [unique axis b]

Click here to get the general positions in text format

No.	(x,y,z) form	Matrix form	Symmetry operation	
			ITA	Seitz
1	x,y,z	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	1	{ 1 | 0 }
2	-x,y+1/2,-z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 (0,1/2,0) 0,y,1/4	{ 2_{010} | 0 1/2 1/2 }
3	-x,-y,-z	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	-1 0,0,0	{ -1 | 0 }
4	x,-y+1/2,z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	c x,1/4,z	{ m_{010} | 0 1/2 1/2 }

ITA data

General positions

4 *e* 1 (1) x,y,z (2) $\bar{x},y+\frac{1}{2},\bar{z}+\frac{1}{2}$ (3) $\bar{x},\bar{y},\bar{z}$ (4) $x,\bar{y}+\frac{1}{2},z+\frac{1}{2}$

Symmetry operations

(1) 1 (2) $2(0,\frac{1}{2},0)$ $0,y,\frac{1}{4}$ (3) $\bar{1}$ $0,0,0$ (4) c $x,\frac{1}{4},z$

SEITZ SYMBOLS FOR SYMMETRY OPERATIONS

Seitz symbols { R | t }

short-hand description of the matrix-column presentations of the symmetry operations of the space groups

rotation (or linear) part R

- specify the type and the order of the symmetry operation;
- orientation of the symmetry element by the direction of the axis for rotations and rotoinversions, or the direction of the normal to reflection planes.

1 and $\bar{1}$	identity and inversion
m	reflections
2, 3, 4 and 6	rotations
$\bar{3}$, $\bar{4}$ and $\bar{6}$	rotoinversions

translation part t

translation parts of the coordinate triplets of the *General position* blocks

EXAMPLE

Seitz symbols for symmetry operations of hexagonal and trigonal crystal systems

ITA description				Seitz symbol
No.	coord. triplet	type	orien-tation	
1)	x, y, z	1		1
2)	$\bar{y}, x-y, z$	3^+	$0,0,z$	3^+_{001}
3)	$\bar{x}+y, \bar{x}, z$	3^-	$0,0,z$	3^-_{001}
4)	$\bar{x}, \bar{y}, z$	2	$0,0,z$	2_{001}
5)	$y, \bar{x}+y, z$	6^-	$0,0,z$	6^-_{001}
6)	$x-y, x, z$	6^+	$0,0,z$	6^+_{001}
7)	$y, x, \bar{z}$	2	$x,x,0$	2_{110}
8)	$x-y, \bar{y}, \bar{z}$	2	$x,0,0$	2_{100}
9)	$\bar{x}, \bar{x}+y, \bar{z}$	2	$0,y,0$	2_{010}
10)	$\bar{y}, \bar{x}, \bar{z}$	2	$x,\bar{x},0$	$2_{1\bar{1}0}$
11)	$\bar{x}+y, y, \bar{z}$	2	$x,2x,0$	2_{120}
12)	$x, x-y, \bar{z}$	2	$2x,x,0$	2_{210}

ITA description				Seitz symbol
No.	coord. triplet	type	orien-tation	
13)	$\bar{x}, \bar{y}, \bar{z}$	$\bar{1}$		$\bar{1}$
14)	$y, \bar{x}+y, \bar{z}$	$\bar{3}^+$	$0,0,z$	$\bar{3}^+_{001}$
15)	$x-y, x, \bar{z}$	$\bar{3}^-$	$0,0,z$	$\bar{3}^-_{001}$
16)	$x, y, \bar{z}$	m	$x,y,0$	m_{001}
17)	$\bar{y}, x-y, \bar{z}$	$\bar{6}^-$	$0,0,z$	$\bar{6}^-_{001}$
18)	$\bar{x}+y, \bar{x}, \bar{z}$	$\bar{6}^+$	$0,0,z$	$\bar{6}^+_{001}$
19)	$\bar{y}, \bar{x}, z$	m	$x,\bar{x},z$	m_{110}
20)	$\bar{x}+y, y, z$	m	$x,2x,z$	m_{100}
21)	$x, x-y, z$	m	$2x,x,z$	m_{010}
22)	y, x, z	m	x,x,z	$m_{1\bar{1}0}$
23)	$x-y, \bar{y}, z$	m	$x,0,z$	m_{120}
24)	$\bar{x}, \bar{x}+y, z$	m	$0,y,z$	m_{210}

Glazer *et al*. Acta Cryst A 70, 300 (2014)

EXAMPLE

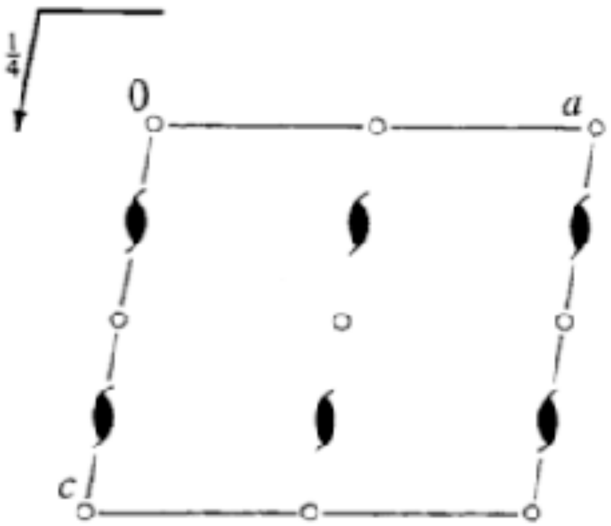
International Tables for Crystallography (2006). Vol. A, Space gro

Space group $P2_1/c$ (No. 14)

$P2_1/c$ C_{2h}^5 $2/m$ I

No. 14 $P12_1/c1$ Patterson sy

UNIQUE AXIS b, CELL CHOICE 1



Generators selected (1); $t(1,0,0)$; $t(0,1,0)$; $t(0,0,1)$; (2); (3)

Positions

Multiplicity,
Wyckoff letter,
Site symmetry

Coordinates

Matrix-column presentation

4 e 1 (1) x,y,z (2) $\bar{x},y+\frac{1}{2},\bar{z}+\frac{1}{2}$ (3) $\bar{x},\bar{y},\bar{z}$ (4) $x,\bar{y}+\frac{1}{2},z+\frac{1}{2}$

Symmetry operations

Geometric interpretation

(1) 1 (2) $2(0,\frac{1}{2},0)$ $0,y,\frac{1}{4}$ (3) $\bar{1}$ $0,0,0$ (4) c $x,\frac{1}{4},z$

Seitz symbols

(1) $\{1|0\}$ (2) $\{2_{010}|0\,1/2\,1/2\}$ (3) $\{\bar{1}|0\}$ (4) $\{m_{010}|0\,1/2\,1/2\}$

Bilbao Crystallographic Server

Problem: Geometric Interpretation of (W,w) — SYMMETRY OPERATION

Symmetry Operation

This program calculates the geometric interpretation of matrix column representation of symmetry operation for a given crystal system or space group.

Input:

i) The crystal system or the space group number.

ii) The matrix column representation of symmetry operation.

If you want to work on a non conventional setting click on **Non conventional setting**, this will show you a form where you have to introduce the transformation matrix relating the conventional setting of the group you have chosen with the non conventional one you are interested in.

Output:

We obtain the geometric interpretation of the symmetry operation.

Introduce the crystal system: monoclinic

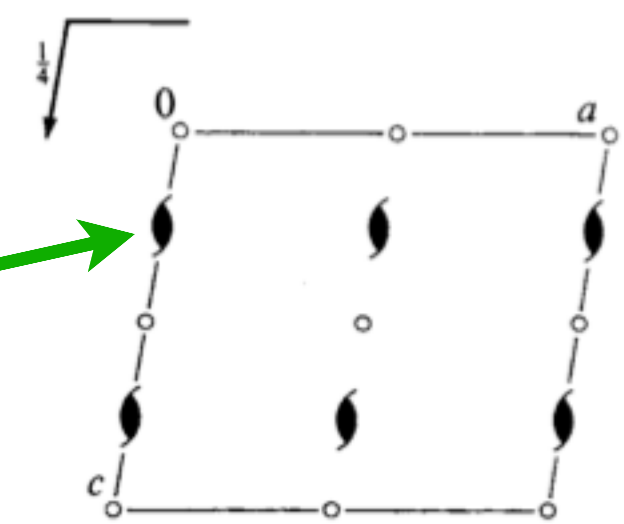
Or enter the sequential number of group as given in the *International Tables for Crystallography*, Vol. A — choose it

Matrix column representation of symmetry operation: -x,y+1/2,-z+1/2

	Rotational part			Translation
In matrix form	1	0	0	0
	0	1	0	0
	0	0	1	0

Standard/Default Setting | Non Conventional Setting | ITA Settings

-x,y+1/2,-z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 (0,1/2,0) 0,y,1/4



EXERCISES

Problem 2.2 (cont.)

Construct the matrix-column pairs (W,w) of the following coordinate triplets:

(1) x,y,z (2) -x,y+1/2,-z+1/2

(3) -x,-y,-z (4) x,-y+1/2, z+1/2

Characterize geometrically these matrix-column pairs taking into account that they refer to a monoclinic basis with unique axis b (type of operation, glide/screw component, location of the symmetry operation).

Use the program SYMMETRY OPERATIONS for the geometric interpretation of the matrix-column pairs of the symmetry operations.

EXERCISES

Problem 2.3

1. Characterize geometrically the matrix-column pairs listed under *General position* of the space group *P4mm* in ITA.

2. Consider the diagram of the symmetry elements of *P4mm*. Try to determine the matrix-column pairs of the symmetry operations whose symmetry elements are indicated on the unit-cell diagram.

3. Compare your results with the results of the program SYMMETRY OPERATIONS

GENERAL
AND
SPECIAL WYCKOFF
POSITIONS
SITE-SYMMETRY

Group Actions

Group Actions

A *group action* of a group $\mathcal{G}$ on a set $\Omega = \{\omega \mid \omega \in \Omega\}$ assigns to each pair (g, ω) an object $\omega' = g(\omega)$ of Ω such that the following hold:

(i) applying two group elements g and g' consecutively has the same effect as applying the product $g'g$, *i.e.* $g'(g(\omega)) = (g'g)(\omega)$

(ii) applying the identity element e of $\mathcal{G}$ has no effect on ω, *i.e.* $e(\omega) = \omega$ for all ω in Ω.

Orbit and Stabilizer

The set $\omega^{\mathcal{G}} := \{g(\omega) \mid g \in \mathcal{G}\}$ of all objects in the orbit of ω is called the *orbit of ω under $\mathcal{G}$*.

The set $S_{\mathcal{G}}(\omega) := \{g \in \mathcal{G} \mid g(\omega) = \omega\}$ of group elements that do not move the object ω is a subgroup of $\mathcal{G}$ called the *stabilizer* of ω in $\mathcal{G}$.

General and special Wyckoff positions

Orbit of a point X_o under G: $G(X_o)=\{(W,w)\ X_o, (W,w)\in G\}$

Multiplicity

Site-symmetry group $S_o=\{(W,w)\}$ of a point X_o

$$(W,w)X_o = X_o$$

$$\left(\begin{array}{ccc|c} a & b & c & w_1 \\ d & e & f & w_2 \\ g & h & i & w_3 \end{array}\right)\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

Multiplicity: $|P|/|S_o|$

General position X_o

$S=\{(I,\mathbf{o})\}\simeq \mathbf{1}$

Multiplicity: $|P|$

Special position X_o

$S>\mathbf{1}=\{(I,\mathbf{o}),...,\}$

Multiplicity: $|P|/|S_o|$

Site-symmetry groups: oriented symbols

General position

(i) coordinate triplets of an image point $\tilde{X}$ of the original point $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under (W,w) of G

-presentation of infinite image points $\tilde{X}$ under the action of (W,w) of G

(ii) short-hand notation of the matrix-column pairs (W,w) of the symmetry operations of G

-presentation of infinite symmetry operations of G

$(W,w) = (I,t_n)(W,w_0),\ 0 \leq w_{i0} < 1$

General Position of Space groups

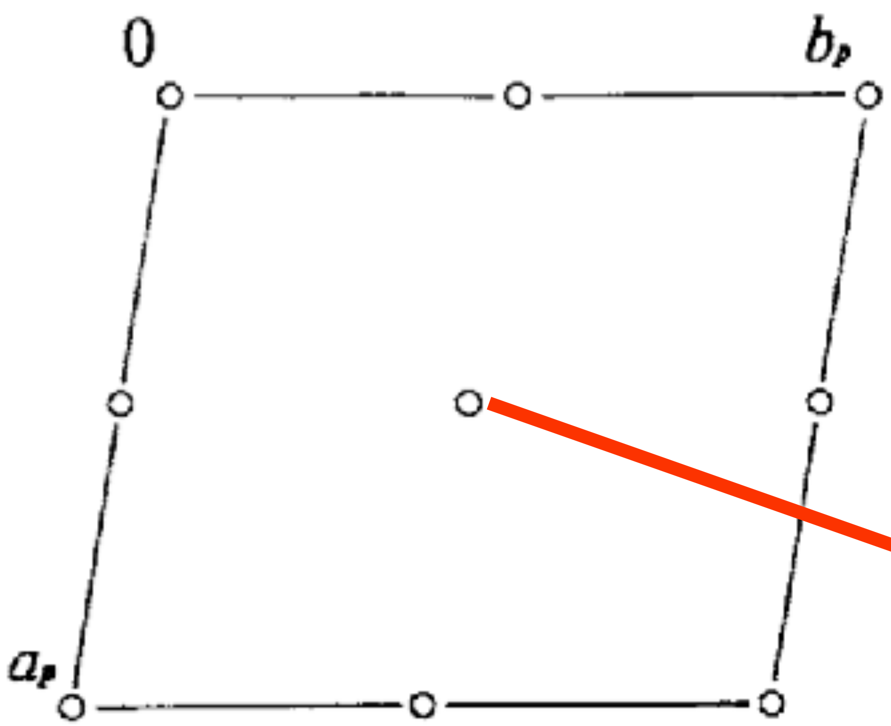
As coordinate triplets of an image point $\tilde{X}$ of the original point $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under (W,w) of G

General position

$(I,0)X$	$(W_2,w_2)X$	...	$(W_m,w_m)X$	...	$(W_i,w_i)X$	
$(I,t_1)X$	$(W_2,w_2+t_1)X$	...	$(W_m,w_m+t_1)X$	...	$(W_i,w_i+t_1)X$	
$(I,t_2)X$	$(W_2,w_2+t_2)X$	...	$(W_m,w_m+t_2)X$	...	$(W_i,w_i+t_2)X$	
...	...	...	...	...	...	
$(I,t_j)X$	$(W_2,w_2+t_j)X$	...	$(W_m,w_m+t_j)X$	...	$(W_i,w_i+t_j)X$	
...	...	...	...	...	...	

Example: Calculation of the Site-symmetry groups

Group P-1



Positions

Multiplicity, Wyckoff letter, Site symmetry			Coordinates	
2	*i*	1	(1) x,y,z	(2) $\bar{x},\bar{y},\bar{z}$
1	*h*	$\bar{1}$	$\frac{1}{2},\frac{1}{2},\frac{1}{2}$	
1	*g*	$\bar{1}$	$0,\frac{1}{2},\frac{1}{2}$	
1	*f*	$\bar{1}$	$\frac{1}{2},0,\frac{1}{2}$	
1	*e*	$\bar{1}$	$\frac{1}{2},\frac{1}{2},0$	
1	*d*	$\bar{1}$	$\frac{1}{2},0,0$	
1	*c*	$\bar{1}$	$0,\frac{1}{2},0$	
1	*b*	$\bar{1}$	$0,0,\frac{1}{2}$	
1	*a*	$\bar{1}$	$0,0,0$	

$$S=\{(W,w),\ (W,w)X_o = X_o\}$$

$$\left(\begin{array}{ccc|c} -1 & & & 0 \\ & -1 & & 0 \\ & & -1 & 0 \end{array}\right)\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

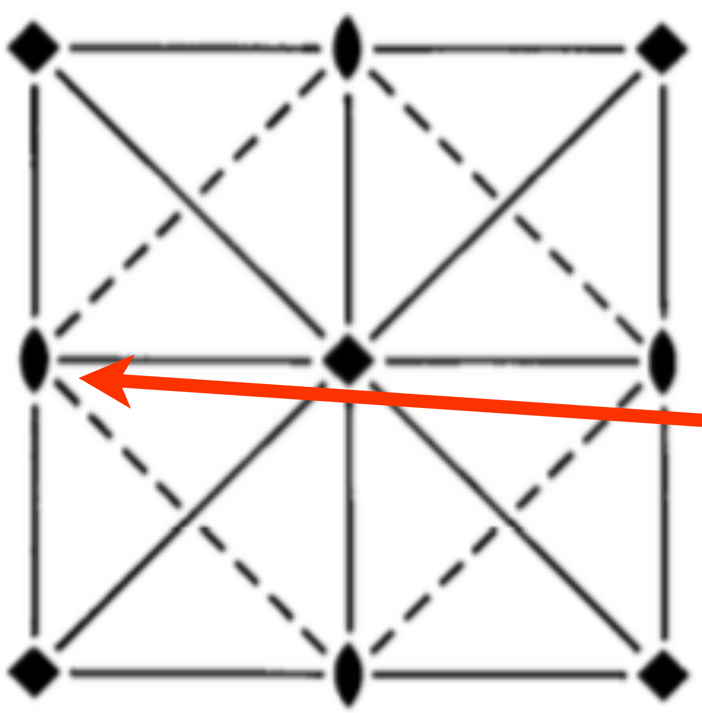
$$S_f=\{(1,0),\ (-1,101)X_f = X_f\}$$

$S_f \simeq \{1, -1\}$ isomorphic

EXERCISES

General and special Wyckoff positions of P4mm

8	g	1	(1) x,y,z	(2) $\bar{x},\bar{y},z$	(3) $\bar{y},x,z$	(4) $y,\bar{x},z$
			(5) $x,\bar{y},z$	(6) $\bar{x},y,z$	(7) $\bar{y},\bar{x},z$	(8) y,x,z
4	f	$.m.$	$x,\frac{1}{2},z$	$\bar{x},\frac{1}{2},z$	$\frac{1}{2},x,z$	$\frac{1}{2},\bar{x},z$
4	e	$.m.$	$x,0,z$	$\bar{x},0,z$	$0,x,z$	$0,\bar{x},z$
4	d	$..m$	x,x,z	$\bar{x},\bar{x},z$	$\bar{x},x,z$	$x,\bar{x},z$
2	c	$2mm.$	$\frac{1}{2},0,z$	$0,\frac{1}{2},z$		
1	b	$4mm$	$\frac{1}{2},\frac{1}{2},z$			
1	a	$4mm$	$0,0,z$			



Symmetry operations

(1) 1 (2) $2\ \ 0,0,z$ (3) $4^{+}\ \ 0,0,z$ (4) $4^{-}\ \ 0,0,z$
(5) $m\ \ x,0,z$ (6) $m\ \ 0,y,z$ (7) $m\ \ x,\bar{x},z$ (8) $m\ \ x,x,z$

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Problem: Wyckoff positions
Site-symmetry groups
Coordinate transformations

WYCKPOS

Wyckoff Positions

space group

How to select the group

The space groups are specified by their number as given in the *International Tables for Crystallography*, Vol. A. You can give this number, if you know it, or you can choose it from the table with the space group numbers and symbols if you click on the link **choose it.**

If you are using this program in the preparation of a paper, please cite it in the following form:

Aroyo, et. al. *Zeitschrift fuer Kristallographie* (2006), 221, 1, 15-27.

Please, enter the sequential number of group as given in *International Tables for Crystallography*, Vol. A or choose it: 68

Standard/Default Setting | Non Conventional Setting | ITA Settings

Standard basis

Transformation of the basis

ITA settings

ITA-Settings for the Space Group 68

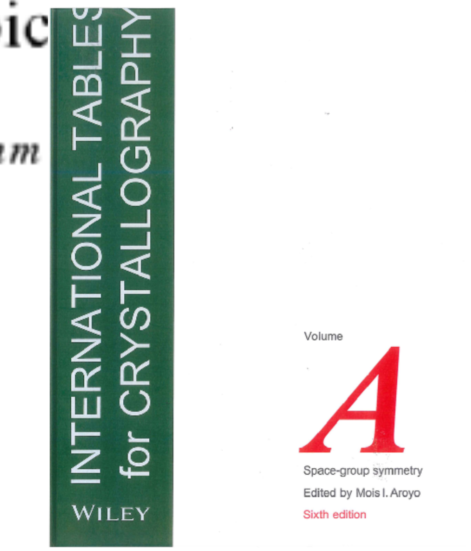
…ces must be read by columns. P is the transformation f…

$(a, b, c)_n = (a, b, c)_s P$

ITA number	Setting	P	P^{-1}
68	*C c c e* [origin 1]	a,b,c	a,b,c
68	*A e a a* [origin 1]	c,a,b	b,c,a
68	*B b e b* [origin 1]	b,c,a	c,a,b
68	*C c c e* [origin 2]	a,b,c	a,b,c
68	*A e a a* [origin 2]	c,a,b	b,c,a
68	*B b e b* [origin 2]	b,c,a	c,a,b

Ccce D_{2h}^{22} *mmm* Orthorhombic

No. 68 $C\ 2/c\ 2/c\ 2/e$ Patterson symmetry *Cmmm*



16	*i*	1	(1) x,y,z (5) $\bar{x},\bar{y},\bar{z}$	(2) $\bar{x}+\frac{1}{2},\bar{y},z$ (6) $x+\frac{1}{2},y,\bar{z}$	(3) $\bar{x},y,\bar{z}+\frac{1}{2}$ (7) $x,\bar{y},z+\frac{1}{2}$	(4) $x+\frac{1}{2},\bar{y},\bar{z}+\frac{1}{2}$ (8) $\bar{x}+\frac{1}{2},y,z+\frac{1}{2}$
8	*h*	..2	$\frac{1}{4},0,z$	$\frac{3}{4},0,\bar{z}+\frac{1}{2}$	$\frac{3}{4},0,\bar{z}$	$\frac{1}{4},0,z+\frac{1}{2}$
8	*g*	..2	$0,\frac{1}{4},z$	$0,\frac{1}{4},\bar{z}+\frac{1}{2}$	$0,\frac{3}{4},\bar{z}$	$0,\frac{3}{4},z+\frac{1}{2}$
8	*f*	.2.	$0,y,\frac{1}{4}$	$\frac{1}{2},\bar{y},\frac{1}{4}$	$0,\bar{y},\frac{3}{4}$	
8	*e*	2..	$x,\frac{1}{4},\frac{1}{4}$	$\bar{x}+\frac{1}{2},\frac{3}{4},\frac{1}{4}$	$\bar{x},\frac{3}{4},\frac{3}{4}$	
8	*d*	$\bar{1}$	$0,0,0$	$\frac{1}{2},0,0$	$0,0,\frac{1}{2}$	
8	*c*	$\bar{1}$	$\frac{1}{4},\frac{3}{4},0$	$\frac{1}{4},\frac{1}{4},0$	$\frac{3}{4},\frac{3}{4},\frac{1}{2}$	
4	*b*	222	$0,\frac{1}{4},\frac{3}{4}$	$0,\frac{3}{4},\frac{1}{4}$		
4	*a*	222	$0,\frac{1}{4},\frac{1}{4}$	$0,\frac{3}{4},\frac{3}{4}$		

Wyckoff Positions of Group 68 (*Ccce*) [origin choice 2]

Multiplicity	Wyckoff letter	Site symmetry	Coordinates (0,0,0) + (1/2,1/2,0) +
16	i	1	(x,y,z) (-x+1/2,-y,z) (-x,y,-z+1/2) (x+1/2,-y,-z+1/2) (-x,-y,-z) (x+1/2,y,-z) (x,-y,z+1/2) (-x+1/2,y,z+1/2)
8	h	..2	(1/4,0,z) (3/4,0,-z+1/2) (3/4,0,-z) (1/4,0,z+1/2)
8	g	..2	(0,1/4,z) (0,1/4,-z+1/2) (0,3/4,-z) (0,3/4,z+1/2)
8	f	.2.	(0,y,1/4) (1/2,-y,1/4) (0,-y,3/4) (1/2,y,3/4)
8	e	2..	(x,1/4,1/4) (-x+1/2,3/4,1/4) (-x,3/4,3/4) (x+1/2,1/4,3/4)
8	d	-1	(0,0,0) (1/2,0,0) (0,0,1/2) (1/2,0,1/2)
8	c	-1	(1/4,3/4,0) (1/4,1/4,0) (3/4,3/4,1/2) (3/4,1/4,1/2)
4	b	222	(0,1/4,3/4) (0,3/4,1/4)
4	a	222	(0,1/4,1/4) (0,3/4,3/4)

Space Group : 68 (*Ccce*) [origin choice 2
Point : (0,1/4,1/4)
Wyckoff Position : 4a

Site Symmetry Group 222

x,y,z	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	
-x,y,-z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 0,y,1/4
-x,-y+1/2,z	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	2 0,1/4,z
x,-y+1/2,-z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 x,1/4,1/4

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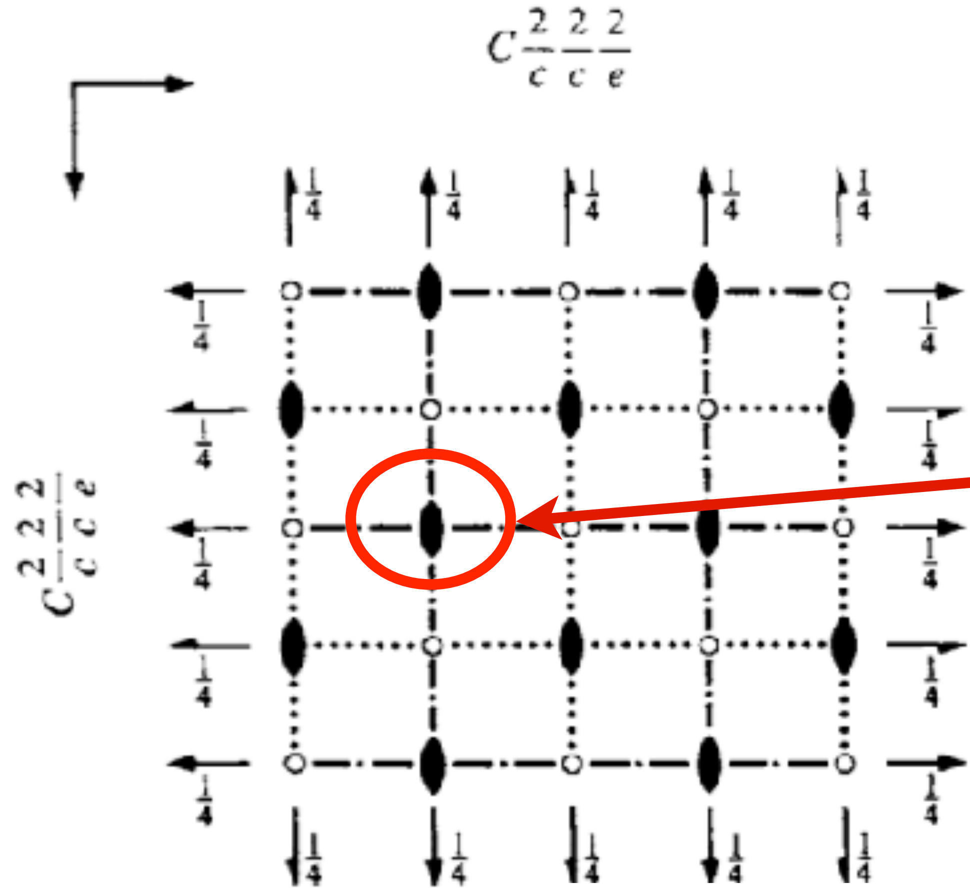
Example WYCKPOS: Wyckoff Positions Ccce (68)



Wyckoff position and site symmetry group of a specific point

Specify the point by its relative coordinates (in fractions or decimals) *Variable parameters (x,y,z) are also accepted*
x = 1/2 y = 1/4 z = 1/4 Show

Space Group : 68 (*Ccce*) [origin choice 2]
Point : (1/2,1/4,1/4)
Wyckoff Position : 4b

Site Symmetry Group 222

x,y,z	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	1
-x+1,y,-z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 1/2,y,1/4
-x+1,-y+1/2,z	$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	2 1/2,1/4,z
x,-y+1/2,-z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 x,1/4,1/4

EXERCISES

Problem 2.4

Consider the special Wyckoff positions of the the space group *P4mm*.

Determine the site-symmetry groups of Wyckoff positions *1a* and *1b*. Compare the results with the listed ITA data

The coordinate triplets (x,1/2,z) and (1/2,x,z), belong to Wyckoff position *4f*. Compare their site-symmetry groups.

Compare your results with the results of the program WYCKPOS.

DOUBLE SPACE GROUPS

Double space groups

Space group $G = \{(R,v)\}$: coset decomposition with respect to T

$$G=(E,0)T+(R_2,v_2)T + \ldots +(R_n,v_n)T$$

The double group ${}^dG$ of G is defined by:

$${}^dG=(E,0)T+(\overline{E},0)T+(R_2,v_2)T+(\overline{R}_2,v_2)T+ \ldots +(R_n,v_n)T+(\overline{R}_n,v_n)T$$

R_i and $\overline{R}_i$ are the elements of the double point group ${}^d\overline{G}$ corresponding to the element R_i of the point group of G, and T is the translation subgroup of G.

Note: $G \not< {}^dG$ — the operations of ${}^dG$ that correspond to **G** do not form a closed set

double translation subgroup ${}^dT$: ${}^dT=(E,0)T+(\overline{E},0)T$

${}^dT \lhd {}^dG$ $\quad$ ${}^dG=(E,0){}^dT+(R_2,v_2){}^dT+ \ldots +(R_n,v_n){}^dT$

T and ${}^dT$: abelian groups $\quad {}^dT=T\otimes\{(E,0),(\overline{E},0)\}$

Double space groups

Action on a vector/point:

$$\bar{R}\mathbf{x} = R\mathbf{x}$$

$$(\bar{R},v)X = (R,v)X$$

Wyckoff positions and site-symmetry groups:

Multiplication rules:

space group G	double space group ${}^dG$
$(R_1,v_1)(R_2,v_2)=(R_1R_2,R_1v_2+v_1)$	$(R_1,v_1)(R_2,v_2)=(R_1R_2,R_1v_2+v_1)$
	$(\bar{R}_1,v_1)(R_2,v_2)=(\bar{R}_1R_2,R_1v_2+v_1)$
	$(R_1,v_1)(\bar{R}_2,v_2)=(R_1\bar{R}_2,R_1v_2+v_1)$
	$(\bar{R}_1,v_1)(\bar{R}_2,v_2)=(\bar{R}_1\bar{R}_2,R_1v_2+v_1)$

DOUBLE CRYSTALLOGRAPHIC GROUPS

FCT/ZTF

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Space-group symmetry

Magnetic Symmetry and Applications

bilbao crystallographic server

ECM31-Oviedo Satellite

...allography online: workshop on
...d applications of the structural
...he Bilbao Crystallographic Serv...

20-21 August 2018

Double point and space groups

DGENPOS	General positions of Double Space groups
REPRESENTATIONS DPG	Irreducible representations of the Double Point Groups
REPRESENTATIONS DSG	Irreducible representations of the Double Space Groups
DSITESYM	Site-symmetry induced representations of Double Space Groups
DCOMPREL	Compatibility relations between the irreducible representations of Double Space Groups
BANDREP	Band representations and Elementary Band representations of Double Space Groups

Subperiodic Groups: Layer, Rod and Frieze Groups

Structure Databases

...lew Article in Nature
...7/2017: Bradlyn *et al.* "Topological quantum
...hemistry" *Nature* (2017). **547**, 298-305.

...lew program: BANDREP
...4/2017: Band representations and Elementary
...and representations of Double Space Groups.

Example DGENPOS

Double space group $P2_12_12_1 (19)$

The symmetry operations are specified by:

- matrix representations
- shorthand notation
  - x,y,z coordinate triplets
  - s_1,s_2 spin components
- Seitz symbols

Symbols of 'double-group' operations

$\bar{E} = {}^d1$

$\bar{R} = {}^d1R = {}^dR$

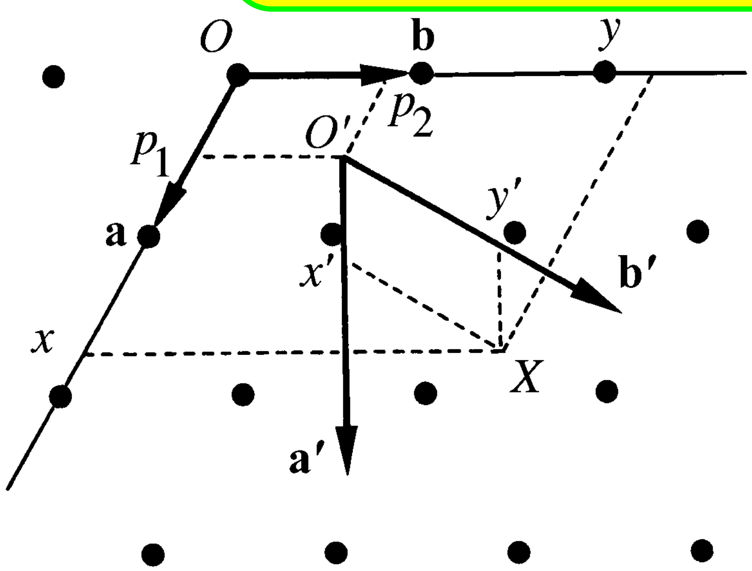
Symmetry operations of the Double Space Group $P2_12_12_1$ (No. 19)

[Get the symmetry operations in plain text format]

N	Shorthand notation	Matrix presentation		Seitz symbol
	(0,0,0)+set			
1	x,y,z s^+,s^-	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	{1|0,0,0}
2	1/2-x,-y,1/2+z $-is^+,is^-$	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\{2_{001}\|1/2,0,1/2\}$
3	-x,1/2+y,1/2-z $-s^-,s^+$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\{2_{010}\|0,1/2,1/2\}$
4	1/2+x,1/2-y,-z $-is^-,-is^+$	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\{2_{100}\|1/2,1/2,0\}$
5	x,y,z $-s^+,-s^-$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\{{}^d1\|0,0,0\}$
6	1/2-x,-y,1/2+z $is^+,-is^-$	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\{{}^d2_{001}\|1/2,0,1/2\}$
7	-x,1/2+y,1/2-z $s^-,-s^+$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\{{}^d2_{010}\|0,1/2,1/2\}$
8	1/2+x,1/2-y,-z is^-,is^+	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\{{}^d2_{100}\|1/2,1/2,0\}$

CO-ORDINATE TRANSFORMATIONS IN CRYSTALLOGRAPHY

Co-ordinate transformations in crystallography



3-dimensional space

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}), \text{ origin O: point X}(x, y, z)$$

$$\downarrow (P,p)$$

$$(\mathbf{a}', \mathbf{b}', \mathbf{c}'), \text{ origin O': point X}(x', y', z')$$

Transformation matrix-column pair (*P*,*p*)

(i) linear part: change of orientation or length:

$$(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (\mathbf{a}, \mathbf{b}, \mathbf{c})\boldsymbol{P}$$

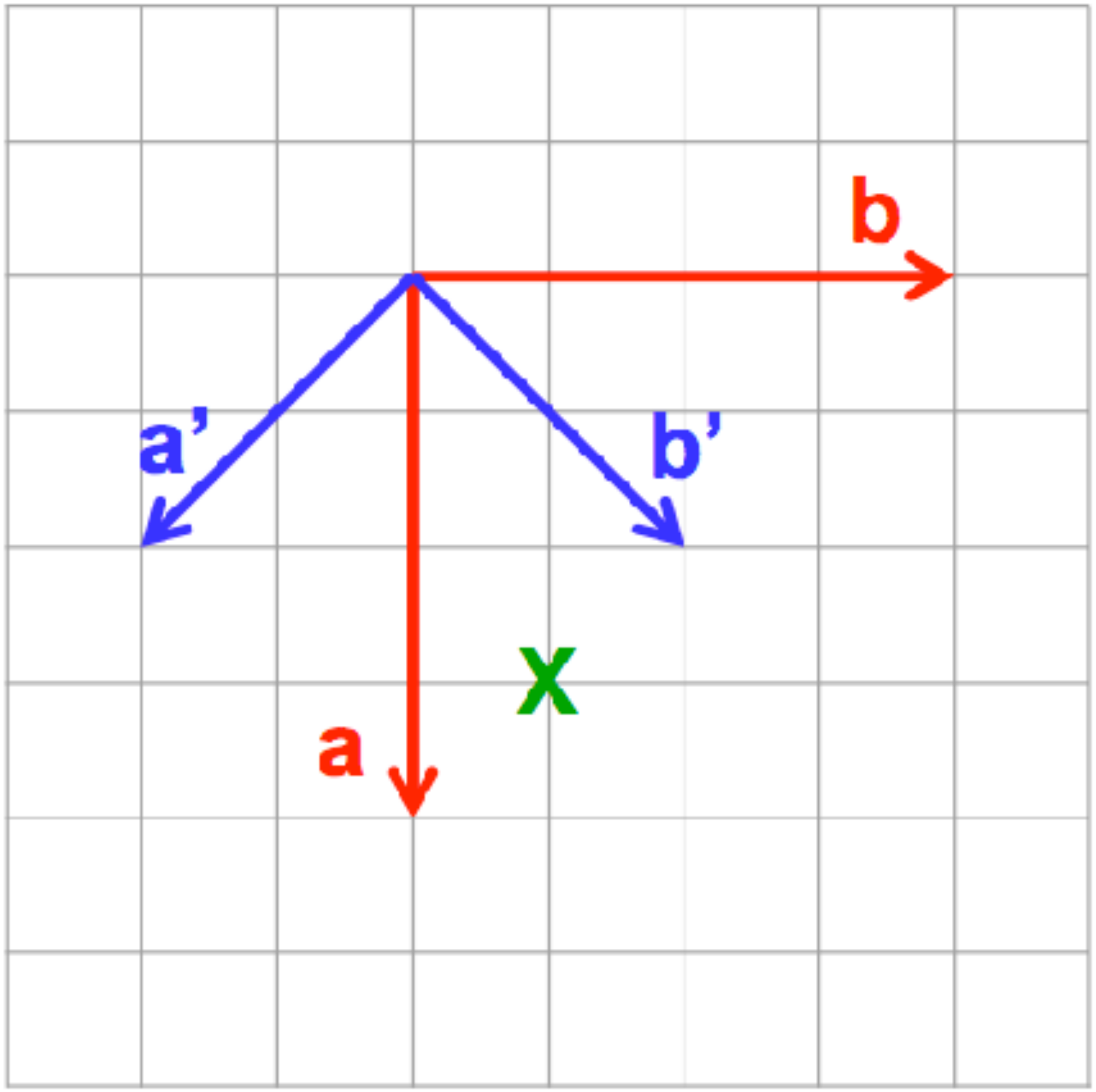
$$= (\mathbf{a}, \mathbf{b}, \mathbf{c}) \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} = (P_{11}\mathbf{a} + P_{21}\mathbf{b} + P_{31}\mathbf{c}, \; P_{12}\mathbf{a} + P_{22}\mathbf{b} + P_{32}\mathbf{c}, \; P_{13}\mathbf{a} + P_{23}\mathbf{b} + P_{33}\mathbf{c}).$$

(ii) origin shift by a shift vector $\boldsymbol{p}(p_1,p_2,p_3)$:

$$O' = O + p$$

the origin *O'* has coordinates (p_1,p_2,p_3) in the old coordinate system

EXAMPLE



$$(\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}') = (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\begin{pmatrix} ? \end{pmatrix}$$

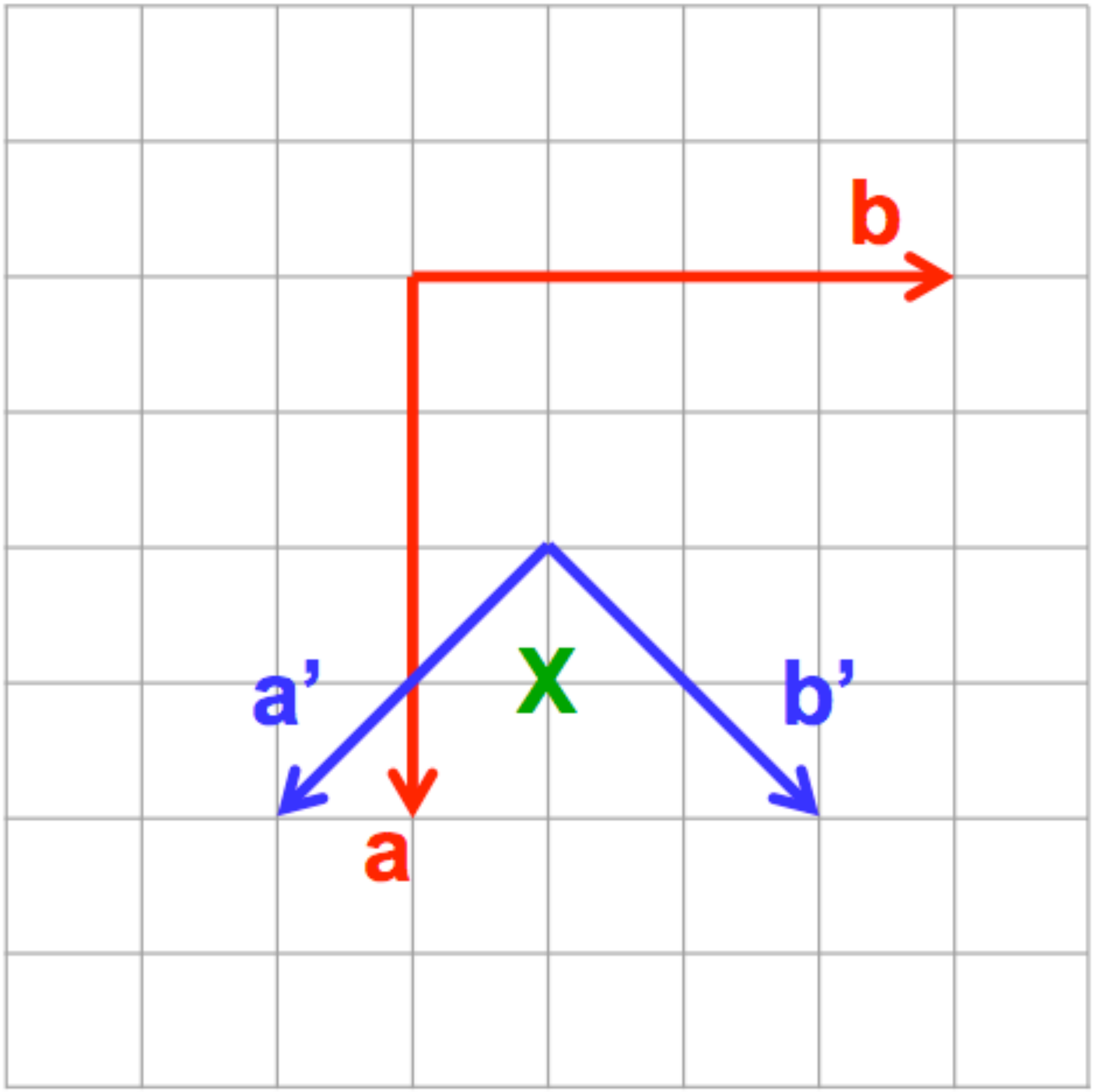
$$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = (\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}')\begin{pmatrix} ? \end{pmatrix}$$

$$\boldsymbol{X} = (3/4, 1/4, 0)$$

$$\boldsymbol{X}' = (\; ? \;)$$

Write "new in terms of old" as column vectors.

EXAMPLE



$p = \begin{pmatrix} ? \end{pmatrix}$

$q = \begin{pmatrix} ? \end{pmatrix}$

$X = (3/4, 1/4, 0)$

$X' = (?)$

Linear parts as before.

Transformation matrix-column pair (P,p)

$$(P,p)=\left(\begin{array}{ccc|c} 1/2 & 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

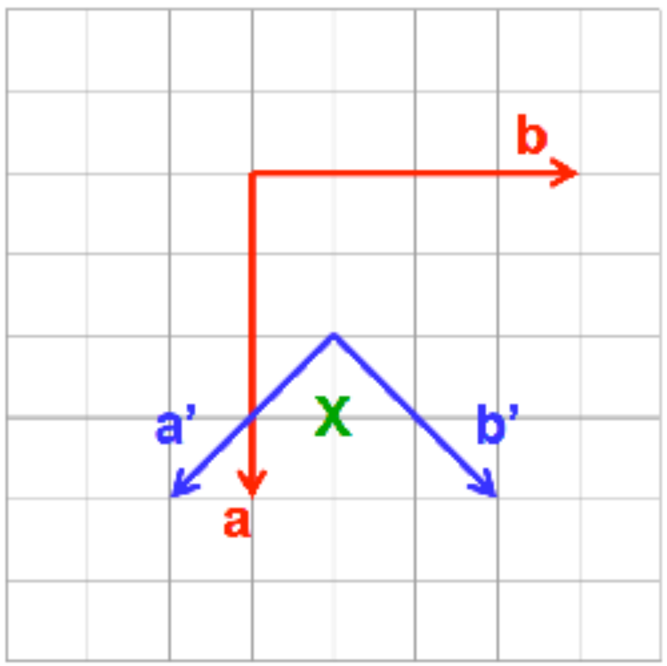
$$(P,p)^{-1}=\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1/4 \\ 1 & 1 & 0 & -3/4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

a'=1/2a-1/2b

b'=1/2a+1/2b

c'=c

$$\mathbf{O'}=\mathbf{O}+\begin{pmatrix} 1/2 \\ 1/4 \\ 0 \end{pmatrix}$$



a=a'+b'

b=-a'+b'

c=c'

$$\mathbf{O}=\mathbf{O'}+\begin{pmatrix} -1/4 \\ -3/4 \\ 0 \end{pmatrix}$$

Co-ordinate transformations in crystallography

Transformation of space-group operations (W,w) by $(\boldsymbol{P},\boldsymbol{p})$:

$$(W',w')=(P,p)^{-1}(W,w)(P,p)$$

Structure-description transformation by $(\boldsymbol{P},\boldsymbol{p})$

unit cell parameters:

metric tensor $\boldsymbol{G}$: $\boldsymbol{G}'=\boldsymbol{P}^t\,\boldsymbol{G}\,\boldsymbol{P}$

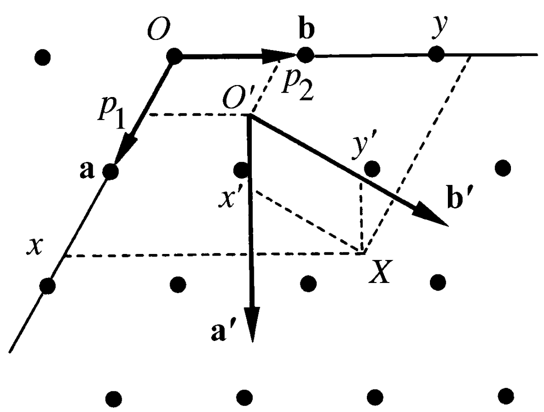
atomic coordinates X(x,y,z):

$$(X')=(P,p)^{-1}(X)$$
$$=(P^{-1},\,-P^{-1}p)(X)$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} P11 & P12 & P13 & p1 \\ P21 & P22 & P23 & p2 \\ P31 & P32 & P33 & p3 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Short-hand notation for the description of transformation matrices

Transformation matrix:



(**a**,**b**,**c**), origin O

$$(P,p)=\begin{pmatrix} P_{11} & P_{12} & P_{13} & p1 \\ P_{21} & P_{22} & P_{23} & p2 \\ P_{31} & P_{32} & P_{33} & p3 \end{pmatrix}$$

(**a'**,**b'**,**c'**), origin O'

notation rules:

-written by columns

-coefficients 0, +1, -1

-different columns in one line

-origin shift

example:

1	-1		-1/4
1	1		-3/4
		1	0

→ { a+b, -a+b, c;-1/4,-3/4,0

EXERCISES

Problem 2.5

The following matrix-column pairs (W,w) are referred with respect to a basis (**a**,**b**,**c**):

(1) x,y,z (2) -x,y+1/2,-z+1/2

(3) -x,-y,-z (4) x,-y+1/2, z+1/2

(i) Determine the corresponding matrix-column pairs (W',w') with respect to the basis (**a'**,**b'**,**c'**)= (**a**,**b**,**c**)**P**, with **P**=**c**,**a**,**b**.

(ii) Determine the coordinates X' of a point X= with respect to the new basis (**a'**,**b'**,**c'**).

0,70
0,31
0,95

Hints

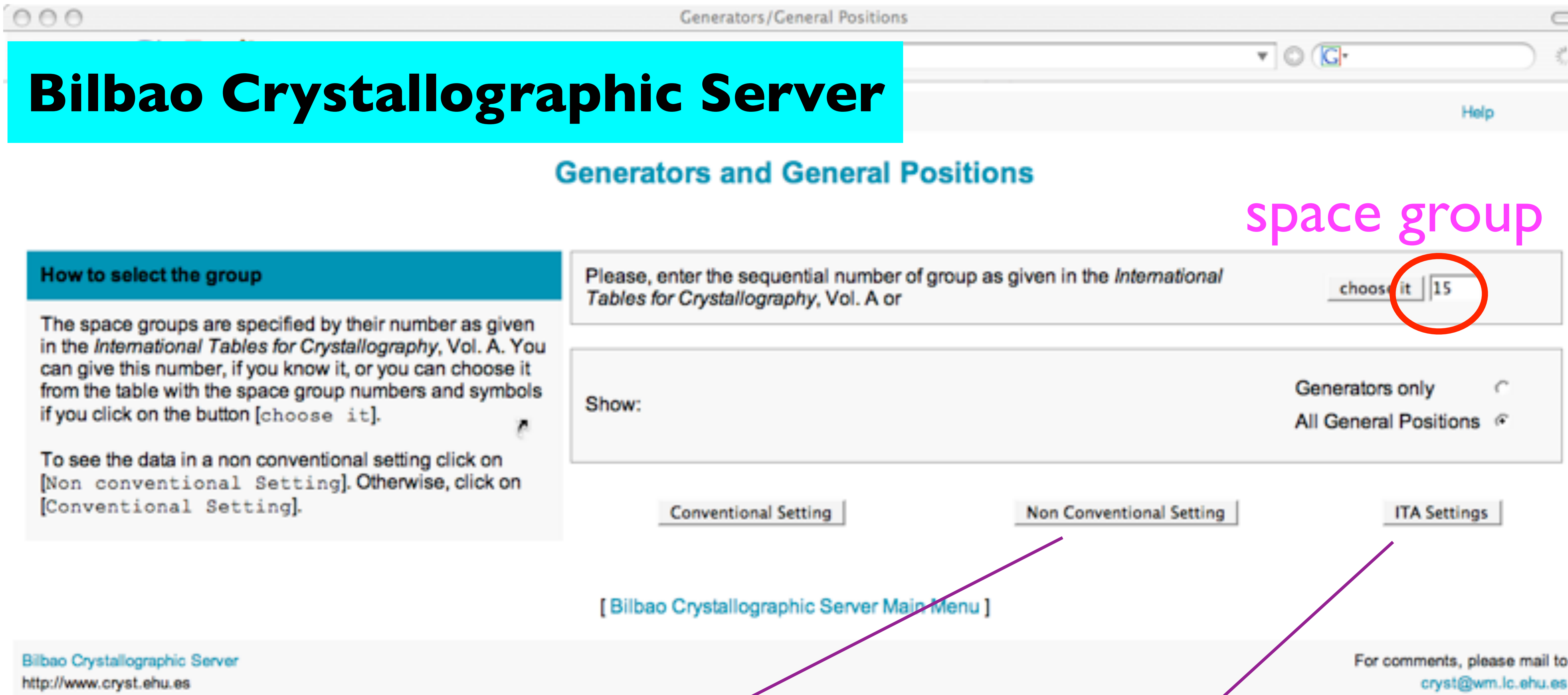
$(W',w')=(P,p)^{-1}(W,w)(P,p)$

$(X')=(P,p)^{-1}(X)$

Problem: Co-ordinate transformations in crystallography

Generators
General positions

GENPOS



Transformation of the basis

ITA-settings symmetry data

ITA-Settings for the Space Group 15

Note: The transformation matrices must be read by columns. **P** is the transformation from standard to the ITA-setting.

$$(\mathbf{a}, \mathbf{b}, \mathbf{c})_n = (\mathbf{a}, \mathbf{b}, \mathbf{c})_s \mathbf{P}$$

Example **GENPOS**:

default setting C12/c1

$(W,w)_{A112/a} = (P,p)^{-1}(W,w)_{C12/c1}(P,p)$

final setting A112/a

ITA number	Setting	P	P^{-1}
15	*C* 1 2/*c* 1	a,b,c	a,b,c
15	*A* 1 2/*n* 1	-a-c,b,a	c,b,-a-c
15	*I* 1 2/*a* 1	c,b,-a-c	-a-c,b,a
15	*A* 1 2/*a* 1	c,-b,a	c,-b,a
15	*C* 1 2/*n* 1	a,-b,-a-c	a,-b,a-c
15	*I* 1 2/*c* 1	-a-c,-b,c	-a-c,-b,c
15	*A* 1 1 2/*a*	c,a,b	b,c,a
15	*B* 1 1 2/*n*	a,-a-c,b	a,c,-a-b
15	*I* 1 1 2/*b*	-a-c,c,b	-a-b,c,b
15	*B* 1 1 2/*b*	a,c,-b	a,-c,b
15	*A* 1 1 2/*n*	-a-c,a,-b	b,-c,-a-b
15	*I* 1 1 2/*a*	c,-a-c,-b	-a-b,-c,a
15	*B* 2/*b* 1 1	b,c,a	c,a,b
15	*C* 2/*n* 1 1	b,a,-a-c	b,a,-b-c
15	*I* 2/*c* 1 1	b,-a-c,c	-b-c,a,c
15	*C* 2/*c* 1 1	-b,a,c	b,-a,c
15	*B* 2/*n* 1 1	-b,-a-c,a	c,-a,-b-c
15	*I* 2/*b* 1 1	-b,c,-a-c	-b-c,-a,b

Example **GENPOS**: ITA settings of C2/c(15)

The general positions of the group 15 (*A* 1 1 2/*a*)

N	Standard/Default Setting *C2/c*			ITA-Setting *A* 1 1 2/*a*		
	(x,y,z) form	matrix form	symmetry operation	(x,y,z) form	matrix form	symmetry operation
1	x, y, z	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	1	x, y, z	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	1
2	-x, y, -z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 0,y,1/4	-x+1/2, -y, z	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	2 1/4,0,z
3	-x, -y, -z	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	-1 0,0,0	-x, -y, -z	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	-1 0,0,0
4	x, -y, z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	c x,0,z	x+1/2, y, -z	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	a x,y,0
5	x+1/2, y+1/2, z	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	t (1/2,1/2,0)	x, y+1/2, z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	t (0,1/2,1/2)
6	-x+1/2, y+1/2, -z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	2 (0,1/2,0) 1/4,y,1/4	-x+1/2, -y+1/2, z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	2 (0,0,1/2) 1/4,1/4,z
7	-x+1/2, -y+1/2, -z	$\begin{pmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	-1 1/4,1/4,0	-x, -y+1/2, -z+1/2	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	-1 0,1/4,1/4
8	x+1/2, -y+1/2, z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$	n (1/2,0,1/2) x,1/4,z	x+1/2, y+1/2, -z+1/2	$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$	n (1/2,1/2,0) x,y,1/4

default setting

***A*112/*a* setting**

EXERCISES

Problem 2.5

Consider the space group $P2_1/c$ (No. 14). Show that the relation between the *General* and *Special* position data of $P112_1/a$ (setting *unique axis c*) can be obtained from the data $P12_1/c1$ (setting *unique axis b*) applying the transformation $(\mathbf{a}',\mathbf{b}',\mathbf{c}')_\mathbf{c} = (\mathbf{a},\mathbf{b},\mathbf{c})_\mathbf{b}\boldsymbol{P}$, with $\boldsymbol{P} = \boldsymbol{c},\boldsymbol{a},\boldsymbol{b}$.

Use the retrieval tools GENPOS (generators and general positions) for accessing the space-group data. Get the data on general positions in different settings either by specifying transformation matrices to new bases, or by selecting one of the 530 settings of the monoclinic and orthorhombic groups listed in *ITA*.

EXERCISES

Problem 2.6

Use the retrieval tools GENPOS or *Generators* and *General positions*, for accessing the space-group data on the *Bilbao Crystallographic Server* or *Symmetry Database* server. Get the data on general and special positions in different settings either by specifying transformation matrices to new bases, or by selecting one of the 530 settings of the monoclinic and orthorhombic groups listed in ITA.

Consider the General position data of the space group *Im-3m* (No. 229). Using the option *Non-conventional setting* obtain the matrix-column pairs of the symmetry operations with respect to a primitive basis, applying the transformation ($\mathbf{a}'$,$\mathbf{b}'$,$\mathbf{c}'$) = 1/2(-$\mathbf{a}$+$\mathbf{b}$+$\mathbf{c}$,$\mathbf{a}$-$\mathbf{b}$+$\mathbf{c}$,$\mathbf{a}$+$\mathbf{b}$-$\mathbf{c}$)