Exercises and References for TMS Satelite Course 2018

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I. PARAMETRIC HAMILTONIANS AND PARALLEL TRANSPORT

1. Consider a parametric family of Hamiltonians $H(\boldsymbol{\lambda})$ with discrete spectrum. Show that the projector $P(\boldsymbol{\lambda})$ onto the N states with energies $\{E_n(\boldsymbol{\lambda})|n=0,1\dots,N\}$ can be written as

$$2\pi i P(\boldsymbol{\lambda}) = \oint_{\mathcal{C}} dz (z - H(\boldsymbol{\lambda}))^{-1}, \tag{1}$$

where z is a complex variable, and $\mathcal{C}$ is a contour enclosing all the $E_n(\boldsymbol{\lambda})$, and no other eigenvalues of $H(\boldsymbol{\lambda})$.

2. Given a Hermitian projector $P(t)$ that depends on some parameter t, show that

$$P\dot{P}P = 0. \tag{2}$$

3. Show that the adiabatic evolution operator $U_A(t)$ satisfies Kato's equation

$$\dot{U_A} = [\dot{P}, P]U_A. \tag{3}$$

4. Given a basis $\{\psi_m(\boldsymbol{\lambda})\}$ of $\mathrm{Im}(P)$, show that the matrix elements of the adiabatic evolution operator $U_A(t)$ can be written as

$$\langle \psi_n(\boldsymbol{\lambda}) | U_A(\boldsymbol{\lambda}(t)) | \psi_m(0)\rangle = W_{nm}(\boldsymbol{\lambda}) \tag{4}$$

where $W_{nm}(\boldsymbol{\lambda})$ is the path-ordered exponential of the Berry connection along the path $\boldsymbol{\lambda}(t)$.

5. Prove directly that

$$P(\boldsymbol{\lambda})U_A(\boldsymbol{\lambda}(t))P(0) = \lim_{\delta\boldsymbol{\lambda}\to 0} P(\boldsymbol{\lambda})P(\boldsymbol{\lambda}-\delta\boldsymbol{\lambda})\dots P(\delta\boldsymbol{\lambda})P(0). \tag{5}$$

II. BERRY PHASE AND WANNIER FUNCTIONS

1. Let

$$\mathfrak{P} = e^{2\pi i \boldsymbol{x}/L}. \tag{6}$$

Show that

$$\mathfrak{P} c_{\boldsymbol{r}} \mathfrak{P}^{-1} = e^{-2\pi i \boldsymbol{r}/L} c_{\boldsymbol{r}}. \tag{7}$$

2. Prove that

$$\sum_{n=1}^{N}\sum_{\boldsymbol{k}} |\psi_{n\boldsymbol{k}}\rangle\langle\psi_{n\boldsymbol{k}}| = \sum_{n=1}^{N}\sum_{\boldsymbol{R}} |w_{n\boldsymbol{R}}\rangle\langle w_{n\boldsymbol{R}}|, \tag{8}$$

where $|\psi_{n\boldsymbol{k}}\rangle$ are the Bloch eigenstates, and $|w_{n\boldsymbol{R}}\rangle$ are the corresponding Wannier functions.

3. Show that in the tight-binding limit in one dimension

$$\log\langle\mathfrak{P}\rangle = -\mathrm{Tr}\left[\oint dk \langle u_{nk}|\partial_k u_{mk}\rangle\right]. \tag{9}$$

4. Recall that the Bloch Hamiltonian for the Rice-Mele chain in terms of the $\sigma = s, p$ basis functions $|\sigma\boldsymbol{R}\rangle$ is

$$h(k) = (\epsilon + 2t\cos k)\sigma_z + 2t\sin k\sigma_y \tag{10}$$

Compute the Wannier functions for this model when a) $t = 0$ and b) $\epsilon = 0$

III. WILSON LOOPS

1. Let $\mathcal{C} = \{\gamma(t), t \in [0,1)\}$ be a curve in the Brillouin zone. Prove that the Wilson line satisfies

$$\mathcal{W}_{\mathcal{C}}^{\dagger} = \mathcal{W}_{\mathcal{C}^{-1}}, \tag{11}$$

where the curve $\mathcal{C}^{-1}$ is defined by the function $\gamma(1-t)$. Hence prove directly that the Wilson line is a unitary operator when restricted to $\mathrm{Im}(P)$.

2. Prove that with Inversion symmetry, that when $\mathbf{k}_\perp \equiv -\mathbf{k}_\perp$ that the eigenvalues of $\mathcal{W}_{\mathbf{g}}(\mathbf{k}_\perp)$ are either real or come in complex conjugate pairs.

3. Let $s = \{\mathcal{R}|0\}$ be a symmetry such that

$$\mathcal{R}\mathbf{k}_\perp \equiv \mathbf{k}_\perp, \tag{12}$$
$$\mathcal{R}\mathbf{g} = \mathbf{g}. \tag{13}$$

Show that in this case the eigenstates of the Wilson loop $\mathcal{W}_{\boldsymbol{g}}(\mathbf{k}_\perp)$ can be labelled by their eigenvalues under the operator $U_{\mathcal{R}}$.

4. Show that for a time-reversal symmetric system that:

   (a) $\mathcal{W}_{\mathbf{g}}(\mathbf{k}_\perp)$ and $\mathcal{W}_{\mathbf{g}}(-\mathbf{k}_\perp)$ are isospectral

   (b) If $\mathcal{T}^2 = -1$ then eigenstates of $\mathcal{W}_{\mathbf{g}}(\mathbf{k}^*)$ at TRIMs $\mathbf{k}^*$ are doubly degenerate.

5. Prove the Ambrose-Singer theorem in the case of a single band (i.e. $\mathrm{rank} P = 1$)

6. Prove for a 1D system that $\det W_{\mathbf{g}} = \pm 1$ is determined by the parity of the occupied states at Γ and X. *Hint*: Use the fact that $I\mathcal{W}_{0\leftarrow -\pi}I^{-1} = \mathcal{W}_{0\leftarrow\pi}$

IV. REFERENCES

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