Lecture notes: Topological phases

Phase transitions generics (informal)

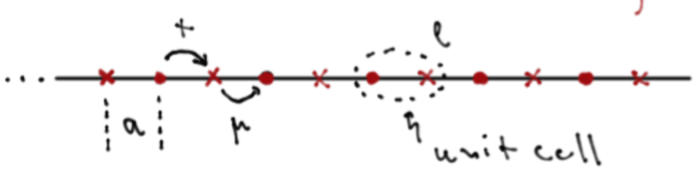
- Phases: distinct macro-states of matter
- distinct phases often distinguished by different local symmetries (cf. ferro-/paramagnetic phase)
- different symmetries characterized by local order parameters
- $\exists$ classical phase transitions and quantum phase transitions (changes in ground states at $T=0$ driven by control parameter)

Q: What happens if topology enters the game?

Will discuss three case studies:

- SSH-chain (a topological insulator)
- BKT-phase transition
- $\mathbb{Z}_2$-spin liquid

The Su-Schrieffer-Haeger (SSH) chain (1978)



(periodic b.c.)

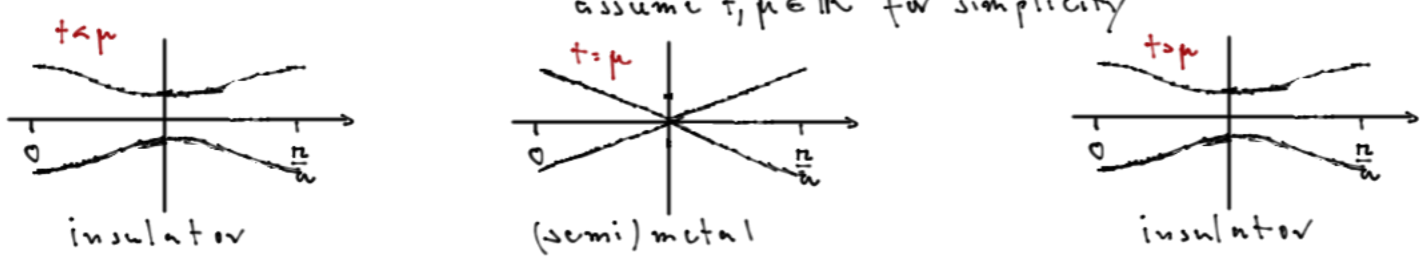
Diagonalize problem in terms of 2-component Bloch wave functions

- $\psi_k(\ell) = \begin{pmatrix} \psi_{\times} \\ \psi_{\bullet} \end{pmatrix}_k e^{ik\ell}$ $\qquad k = 0, \frac{2\pi}{L}, \dots, \frac{\pi}{a}$ $\qquad L = N(2a) =$ chain length

- $\hat{H}_k = \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix} \begin{matrix} \times \\ \bullet \end{matrix}$ $\qquad q_k = t + e^{ik\cdot 2a}\mu$

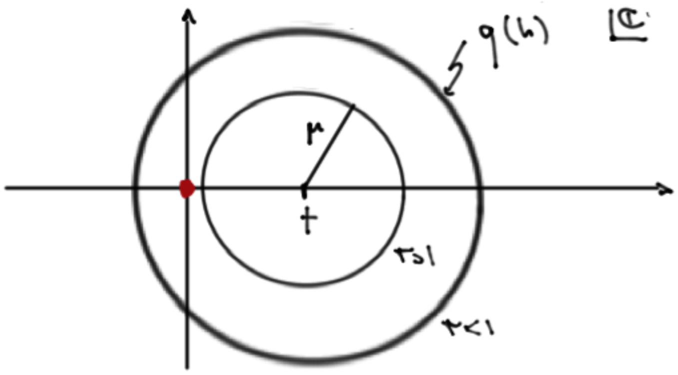
- Eigenvalues: $\epsilon_k^{\pm} = \pm |q_k| = \left(t^2 + \mu^2 + 2t\mu \cos(2ka)\right)^{1/2}$

assume $t, \mu \in \mathbb{R}$ for simplicity



• interpretation: $r \equiv \frac{t}{\mu} = 1$ marks quantum phase transition between two distinct insulating phases. No change in symmetry, no local order parameter.

• topological order: ground states of $r<1$ and $r>1$ carry distinct topological invariant



• curve $S^1 \to \mathbb{C}\setminus\{0\}$

$k \mapsto q(k)$

homotopically trivial/non-trivial for $r>1$/$r<1$

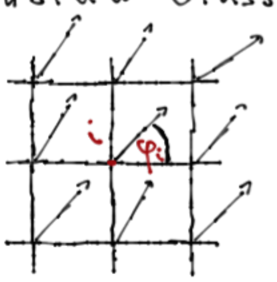
• observable difference: cut system open depending on

the system with $r>1$ has/has not two zero energy boundary states sharply ($\mathcal{O}(a)$) localized at system boundaries. The insulating $r>1$ phase has a 'conducting surface'.

• Summary: $\exists$ (second order) topological quantum phase transitions without symmetry changes/local order parameter.

The Berezinsky-Kosterlitz-Thouless (BKT) transition

• Consider classical xy-model in 2d



$$Z = \int \underbrace{\mathcal{D}\varphi}_{\prod_i d\varphi_i} \exp\Big(-J \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j)\Big) \qquad J = \overset{\text{ferro. exchange}}{J_f}/T > 0$$

• Mermin-Wagner Thm.: no symmetry breaking phase transition

... and yet the model has a phase transition

• Consider spin-spin correlation function $C_{i,j} = \langle e^{i\varphi_i} e^{-i\varphi_j} \rangle$

1) Low T: $J \gg 1$

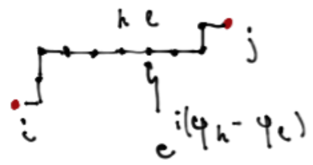
$$S[\varphi] \equiv J \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j) \simeq -\frac{J}{2} \int d^2x \, \nabla\varphi \nabla\varphi + \text{const.}$$

$$\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle = \ln\Big(|x_i - x_j| / \underbrace{a}_{\text{lattice spacing}}\Big) / \pi J$$

$$C_{i,j} = e^{-\frac{1}{2}\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle} = \left(\frac{a}{|x_i - x_j|}\right)^{1/2\pi J}$$

$\rightsquigarrow$ power law correlations

2) High T, $\xi \ll 1$ ~ Expand action in ξ



$$C_{ij} \sim \xi^{d_{ij}} \sim e^{-\ln\xi \cdot c \cdot |x_i - x_j|}$$

d_{ij}: Manhattan metric

exponentially decaying cov.

Conclusion: There must be a finite T phase transition. What discriminates 1) from 2)?: 2) accounts for phase windings of compact var. φ.

Strategy: Teach 1) to include windings

Vortices

- cannot be individually destroyed, however
- vortices ('charges') and anti-vortices (anti-charges) may annihilate each other.
- Individual vortex at r_0 described by, e.g., $\varphi(r) = \tan^{-1}\left(\frac{(r-r_0)_2}{(r-r_0)_1}\right) + \frac{\pi}{2}$ + small fluctuations $(|r-r_0| \gtrsim a)$
- have action cost $(r_0 = 0)$: • $\nabla\varphi = \frac{1}{r} e_\varphi$

$$S_v = \frac{\xi}{2} \int_a^L r\,dr \int_0^{2\pi} d\varphi \frac{1}{r^2} + S^{UV}(a) =$$

$S^{UV}(a)$: short distance action

$$= \pi \xi \ln\left(\frac{L}{a}\right) + S^{UV}(a)$$

~ A single vortex costs action ~ ln L

- Q: Will vortices be present in the system?

Free energy of a vortex: $F_V = -T \ln Z_V = -T \ln e^{-S_V} \times \left(\frac{L}{a}\right)^2$

$\left(\frac{L}{a}\right)^2$: # of different vortex center coord.

$$= \ln\left(\frac{L}{a}\right) \times \{\pi \xi_f - T 2\} + \text{const.}$$

$\pi\xi_f$: vortex energy; $T2$: entropy

A: For $T \gtrsim T_{BKT} = \frac{\pi}{2}\xi_f$ vortex formation becomes favorable.

• Q: What happens in high temperature phase?

A: Estimate action of vortex anti-vortex pair

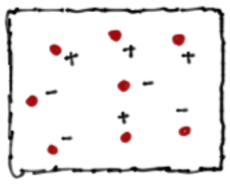
$$S(+,x_1;-,x_2) = 2S^{UV}(a) + 2\pi\zeta \ln\left(|x_1-x_2|/a\right)$$

cf. action of two particles in 2d with electric charge $\pm q(a) \equiv \pm 2\pi\zeta$ and fugacity $y(a) \equiv \exp\left(-S^{UV}(a)\right)$

1) low $T < T_{BKT}$

thermal fluctuations: tightly bound dipoles

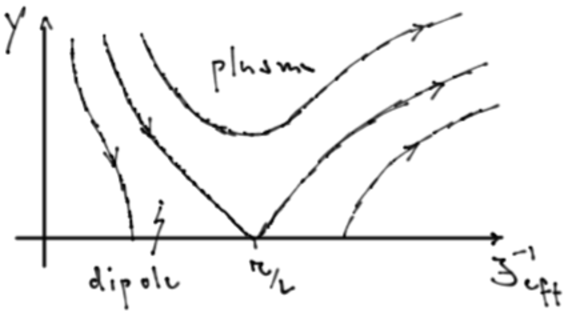
2) high $T > T_{BKT}$



plasma of charged particles

• In depth analysis: two scaling variables $q(a) \sim \zeta_{eff}(a)$, $y(a)$. These two flow (depend on) cutoff a. At lower resolution scales screening

leads to 'renormalization' →



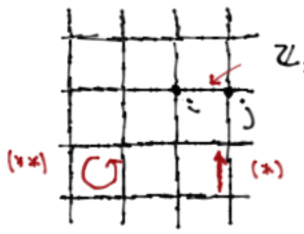
• Summary:
  - topological defects may behave like particles and
  - interact
  - They may drive a finite T classical phase transition
  - without symmetry breaking

$\mathbb{Z}_2$-lattice gauge theory (Wegner 71, review Kogut RMP 79)

- $\mathbb{Z}_2$- degrees of freedom frequently emerging in correlated fermion systems (Senthil & Fisher, PRB 62, 7850 (2000)): $\underbrace{c^\dagger}_{\text{fermion}} \to \underbrace{e^{i\phi}}_{\text{charge}} \cdot \underbrace{f^\dagger}_{\text{neutral fermion}} = e^{i\phi}(-1)\times(-1)\, f^\dagger$

  ⇝ an emergent 'gauge degree of freedom'

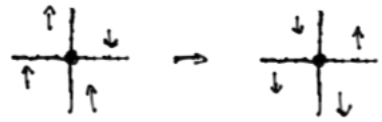
- Formulate $\mathbb{Z}_2$ gauge theory on a lattice. (Here 2d □-lattice for simplicity)



$\mathbb{Z}_2$- d.o.f. $|\pm\rangle_{ij}$ eigenstates of op $\sigma_{z,ij}$.

- local gauge trafo at i acts through $\prod_{\text{n.n. } i} \sigma_{x,ij}$

  ⇝ $|\pm\rangle_{ij} \to |\mp\rangle_{ij}$



- dynamical players of $\mathbb{Z}_2$ lattice gauge theory
  - gauge field along link $i \to j$: $\sigma_{z,ij}$ $\left(\hat{=}\ e^{iA_{ij}} \text{ in } U(1) \text{ lattice ED}\right)$
  - electric flux through link $i \to j$: $\sigma_{x,ij}^{(*)}$. $\quad \sigma_z \sigma_x \sigma_z = -\sigma_x \quad \left(\hat{=}\ e^{iA} E e^{-iA} = E+1\right)$
  - magnetic flux through plaquette s: $\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}^{(**)}$. $\left(\hat{=}\ e^{iA_{ij}} \cdots e^{iA_{li}}\right)$

  electric and magnetic flux are gauge invariant

- gauge invariant Hamiltonian

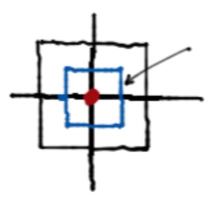
$$\hat{H} = -\gamma \sum_{\text{links}} \sigma_{x,ij}^{(*)} - \lambda \sum_{\square} \sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}^{(**)}$$

- system supports quantum phase transition driven by $t \equiv \lambda/\gamma$

  1) confining phase $t \leq 1$
  2) topological phase $t \geq 1$

1) Confining phase

- Define 'charge density op': $\hat{\rho}_i = \prod_j \sigma_{x,ij}$



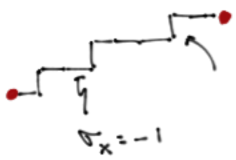
  Charge is locally conserved by $\hat{H}$: $[\hat{H}, \hat{\rho}_i] = 0$

- Ground state in charge neutral sector : $\forall i \quad \hat{\rho}_i |\Psi\rangle = 0$:

  $\sigma_{x,ij} = 1$ globally

• Q: What is ground state in sector of Hilbert space with two charges sitting at i,j?

A: $(\lambda = 0)$



a minimal electric flux line. Costs energy $2\gamma\, d(i,j) = \sigma\, d(i,j)$

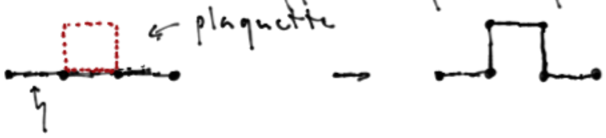
Manhattan

string tension

Energy grows linearly with distance: confinement

• Q: What is the effect of the flux term on this

A:



$\sim$ string fluctuations $\sigma \to 2\gamma - \lambda^2/4\gamma$

strength of pert.

excitation energy

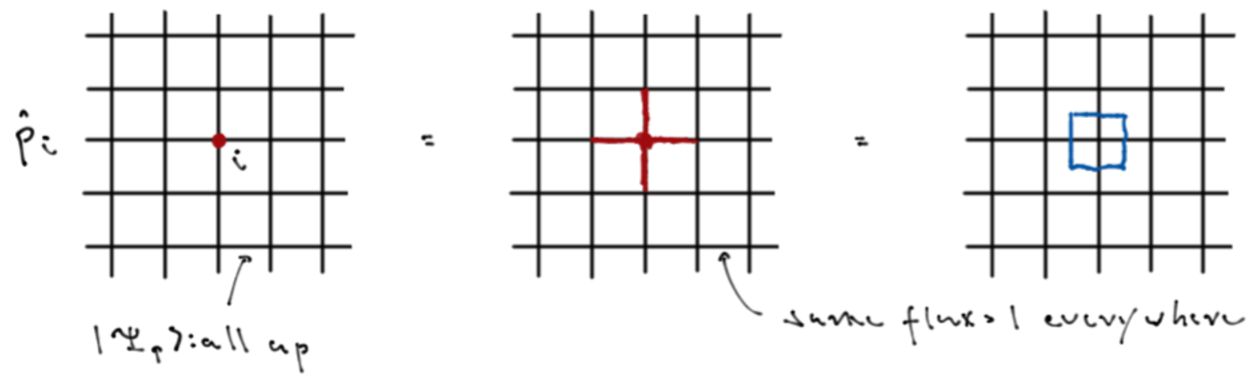
magnetic flux term softens string tension

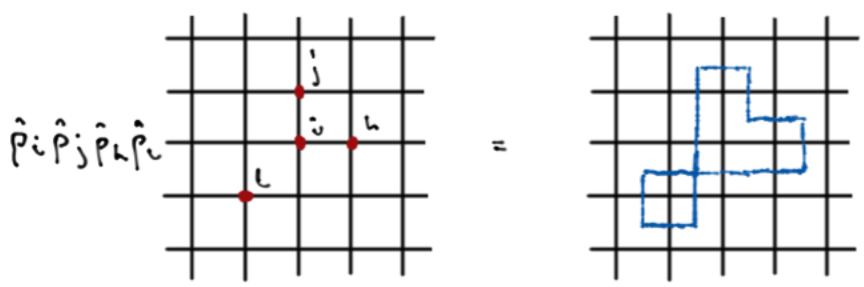
2) Spin liquid phase

• Consider limit $\gamma = 0$. Ground state has flux $\prod_{\square} \sigma_z \cdots \sigma_z$ on each plaquette. This is an implicit characterization.

• Q: How does the ground state in a sector of fixed charge look like?

A: Start from all spins $\sigma_{z,ij} = 1$ state $|\Psi_f\rangle$. This is not a charge eigenstate





ground state: state of all equal weight superposition of closed strings (cf. BCS), a string net condensate

Corrolary: charges totally deconfined.

Q: Is the ground state unique?

A: Def.: V: no. of vertices, E: no. of edges, F: no. of faces

Compactify surface (for simplicity), e.g. . Counting:

E d.o.f. (the spins)

−(V−1) charge constraints (−1 is overall charge neutrality)

−(F−1) flux constraint (−1 is overall flux neutrality)

$\underbrace{-F+E-V}$ -2 qubit d.o.f. remain

(−) Euler-Characteristic $\chi = 2-2g$. Surface genus g.

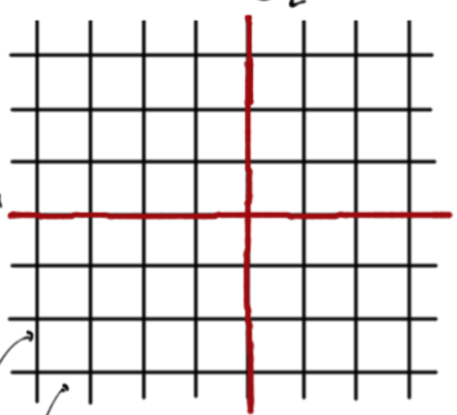
~ ground state degeneracy: 2^{2g}. A hallmark of topological matter.

Q: How do we characterize different ground states?

A:

C_2

C_1



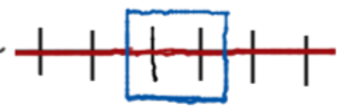
assume periodic boundary conditions

a torus, T^2

$\sigma_z^a = \prod_{C_a} \sigma_{z,ij}$.

or any other curve winding around T^2.

Claim: $\{\sigma_z^1, \sigma_z^2\}$ are topological invariants of each charge sector. E.g.

 and 

same C'

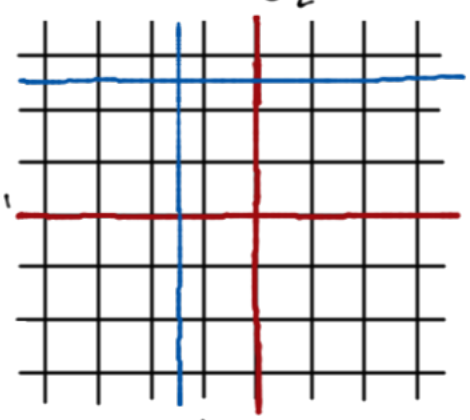
Invariants changed by nonlocal operators.

$\sigma_x^a = \prod_{C'_a} \sigma_{x,ij}$ $\qquad [\sigma_x^a, \sigma_z^a]_+ = 0$

C_2

C'_2

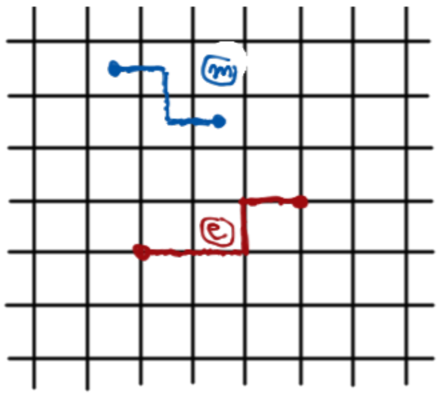
C_1



C'_1

Global qubits $\{\sigma_x^a, \sigma_z^a\}$ provide topological characterization of ground state ~ topological quantum computation.

Q: What are excitations of the system?

A: 

m: a 'magnetic' excitation $= \prod_{\substack{\text{all links} \\ \text{cut by string}}} \sigma_{x,ij}$

changes flux of terminal plaquettes. Costs energy 2λ.

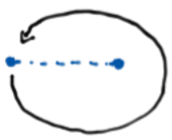
e: an 'electric excitation $= \prod_{\substack{\text{all links along} \\ \text{path}}} \sigma_{z,ij}$

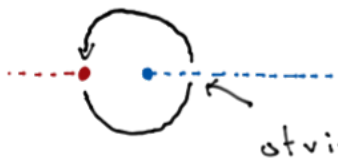
changes charge at terminal points. If system contains 'chemical potential' $\mu \cdot \sum_i \hat{p}_i$, energy cost 2μ.

Think of e, m as quantum particles forming on top of (closed) string ground state.

Q: What is the statistics of these particles?

A: Find out what happens as we braid them around one other.

 $\rightsquigarrow$ nothing happens  : a system of bosons. The same with .

However:  $= (-)$

string of σ_x 'cuts' through one of σ_z: a minus sign

 are semions relative to each other. Note: particle exchange: a π-rotation

	π-exch.	2π-braid
fermions	-1	1
semions	i	-1

Something interesting happens if we fuse m and e into a composite excitation:

 are fermions relative to each other. Have generated fermions

as
- emergent particles of gauge theory
- terminal excitations of strings (cf. Jordan-Wigner in (d))
- particles obeying strict parity conservation

$\rightsquigarrow$ conceptual proximity to string theory

- Summary:
  - $\mathbb{Z}_2$ gauge theory has phase transition without local order parameter or symmetry breaking
  - $\mathbb{Z}_2$ topological fluid = a string condensate
  - Rigorous ($L\to\infty$) ground state degeneracy 2^3 (the closest approx. to an 'order parameter')
  - Supports (abelian) quasi particles as excitations, including emergent fermions
- Appendix: U(1) lattice gauge theory (ideas)

Put electrodynamics on a 3+1 dim. lattice. Ingredients (Euclidean metric)

- gauge potential A_{ij} sits on links and enters through phases $e^{iA_{ij}}$
- gauge transformation ϕ_i acts on nodes via phases $e^{i\phi_i}$
- gauge invariant action: $S[A] \sim \sum_{\square} e^{iA_{ij}} e^{iA_{jl}} e^{iA_{lk}} e^{iA_{ki}}$

  becomes $S_e[A] = \frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\nu\mu}$ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in continuum limit (and passage to Minkowski)
- Hamiltonian of lattice gauge theory

  Chose gauge $A_0 = \varphi = 0$. $S_e[A] = \frac{1}{2}\int d^4x \left(\partial_t A_i \partial_t A_i - (\nabla\times A)_i (\nabla\times A_i)\right) =$

  $$\left(= \frac{1}{2}\int d^4x \left(E_i E_i - B_i B_i \right)\right)$$

  Canonical momentum (of 'coordinate' A) $\delta S_e[A]/\delta \partial_t A_i = E_i$

  $\rightsquigarrow$ Hamiltonian $H_e[A,E] = \frac{1}{2}\int d^4x \left(E_i^2 + B_i^2 \right) \rightsquigarrow$ on a 3d (!) lattice:

  $$H = \sum_{\langle i,j\rangle} \hat{E}_{ij}^2 + \sum_{\square} e^{i\hat{A}_{ij}} \cdots e^{i\hat{A}_{ki}} \qquad [\hat{A}_{ij}, \hat{E}_{ij}] = 1$$

  this is the U(1) analog of the $\mathbb{Z}_2$ - Hamiltonian discussed above.