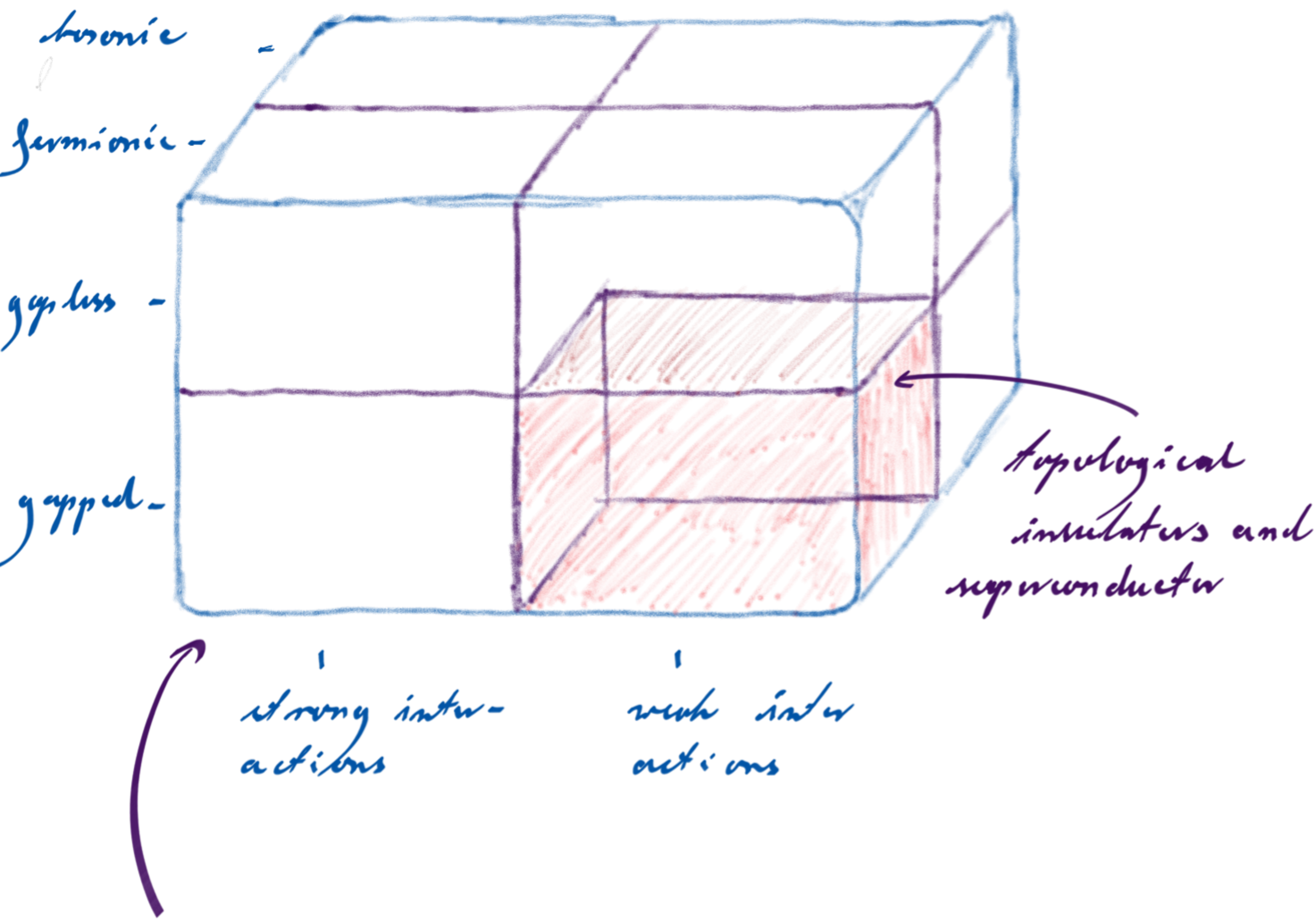
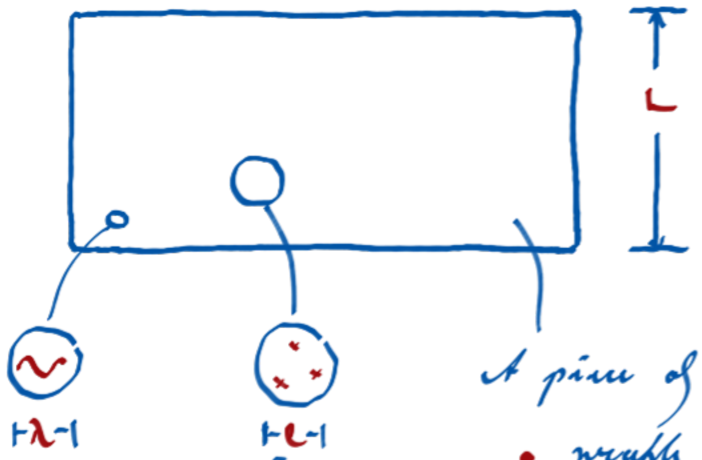


Topological Matter

Symmetries and Topology in Non-Interacting Fermion Systems

The Setting:



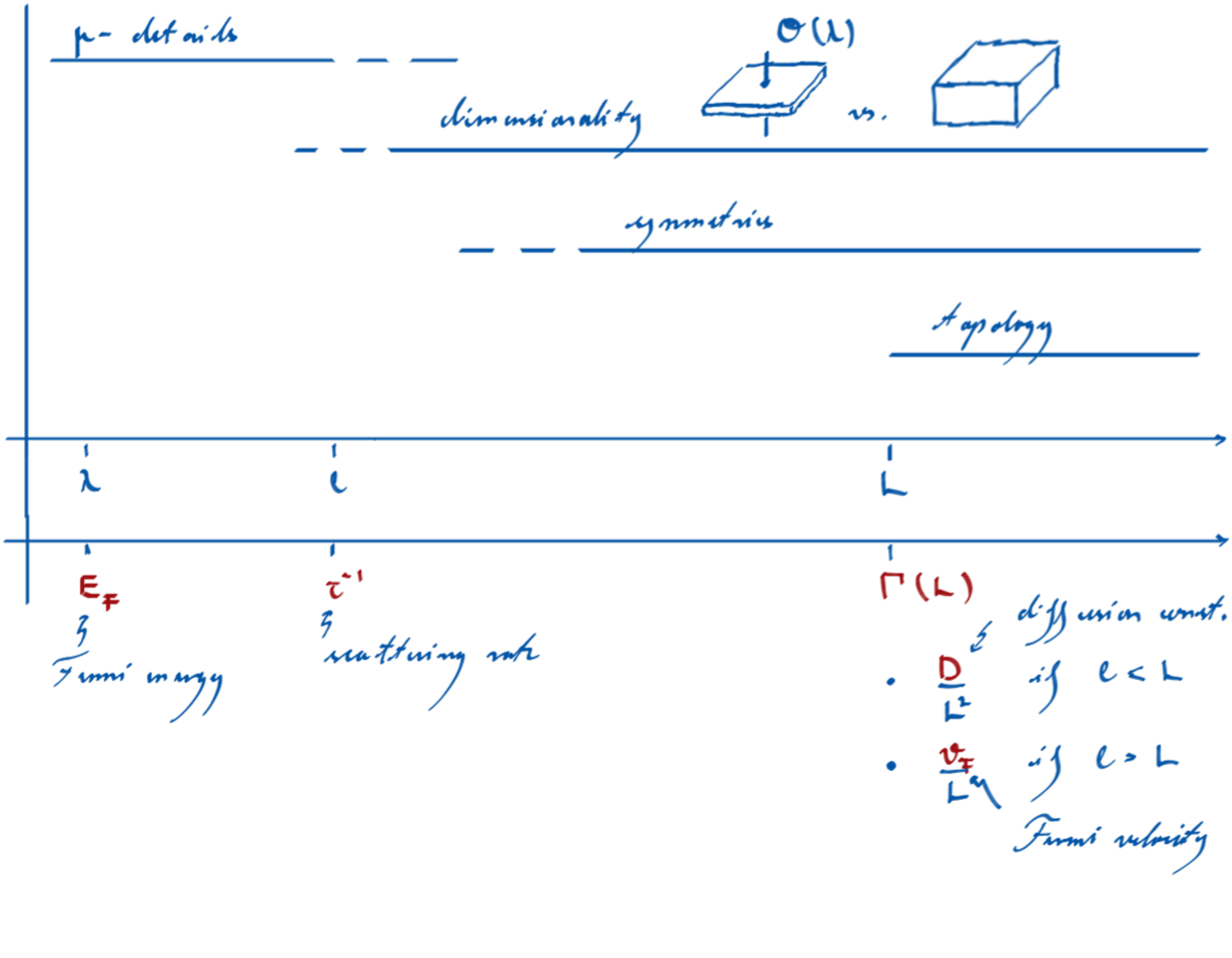
$\lambda \ll \ell \ll L$

A piece of

- weakly interacting
- non translationally invariant

superconductor / metal / insulator

Physical concepts relevant to different length scales



Symmetries in QM

Given: Hilbert Space $\mathcal{H}$ of Dimension N States $|\psi\rangle$

Hamiltonian $\hat{H}$ Symmetry group $G \ni g$

Symmetry group represented on $\mathcal{H}$ through transformations $|\psi\rangle \to g|\psi\rangle$

Many symmetries of QM (Translations, rotations, crystal point operations, ...) are unitary symmetries $\equiv g \in U(N)$, the group of unitary transformations of $\mathcal{H}$.

More important to present content: anti-unitary symmetries when G is represented through anti-unitary maps $g \equiv \Theta$.

Reminder: $\Theta: \mathcal{H} \to \mathcal{H}$ is anti-unitary iff

- $\langle \Theta\psi, \Theta\psi' \rangle = \overline{\langle \psi, \psi' \rangle} = \langle \psi', \psi \rangle$
- $\Theta|z\psi\rangle = \bar{z}\, \Theta|\psi\rangle \quad z \in \mathbb{C}$

$\exists\, U \in U(N): \quad \Theta = UK \qquad K$: complex conjugation

Physics: $\Theta^2 = \pm\, \mathrm{id}$ e.g.: $K^2 = \mathrm{id}$, $\left[\begin{pmatrix} & -i \\ i & \end{pmatrix} K\right]^2 = -\,\mathrm{id}$.

Examples: Time reversal, particle-hole symmetry, charge conjugation symmetry, ...

Warning: Highly confusing denotations / definitions in circulation!

10 Symmetry classes (nutshell intro)

Def.: A system is *time reversal symmetric* if $\exists\, \Theta_T :\ \Theta_T \hat{H} \Theta_T^{-1} = +\hat{H}$

$\leadsto$ 3 possibilities:

sym.	Θ_T^2	T
$-$	x	0
$+$	+id	+1
$+$	-id	-1

Comment: Often (but not always*) systems with ½-integer spin have $T=-1$

Def.: A system is *charge conjugation symmetric* iff $\exists\, \Theta_C :\ \Theta_C \hat{H} \Theta_C^{-1} = -\hat{H}$

$\leadsto$ 3 possibilities

sym.	Θ_C^2	C
$-$	x	0
$+$	+id.	+1
$+$	-id	-1

Comment: Usually** requires Nambu structure: $H = \begin{pmatrix} h & \Delta \\ \bar{\Delta} & -h^T \end{pmatrix}$ symmetric under $\Theta_C = \begin{pmatrix} & -i \\ i & \end{pmatrix} K$.

Def.: A system is called *chiral-symmetric* if it is symmetric under $S = C \circ T$ (and therefore also $T\circ C = \underbrace{T \circ C \circ T^{-1}}_{C'} \circ T$)

$\leadsto$ 4 (!) possibilities. Comments

T	C	S
±1	±1	1
±1	0	0
0	±1	0
0	0	0
0	0	1

- $(\Theta_C \cdot \Theta_T)\hat{H} \cdot (\Theta_T^{-1}\cdot\Theta_C^{-1}) = -\hat{H} \leadsto S=1$: $\hat{H}$ anti-commutes with unitary/ unitary operator, a *chiral symmetry*
- $T=0,\ C=0$ does not fix S

In total $\underset{T}{3} \times \underset{C}{3} + 1 = 10$ *symmetry classes*

---

* Example: spinless wave functions of graphene subject to intervalley scattering have $T=-1$. $\hat{H}_{\text{node}} = \begin{pmatrix} v & p_x+ip_y \\ p_x-ip_y & v \end{pmatrix} \leadsto \sigma_y \hat{H}^T_{\text{node}} \sigma_y = \hat{H}_{\text{node}}$

** Example: $\hat{H} = i\hat{A},\ \hat{A}^T = -\hat{A}$ C-symmetric under $\Theta_C = K$

• General remarks on symmetries

• A system may exhibit symmetry under combination of anti-unitary and unitary symmetries. Example (Berry - Robnik '86)

• $R|x,y\rangle = |-x,y\rangle$

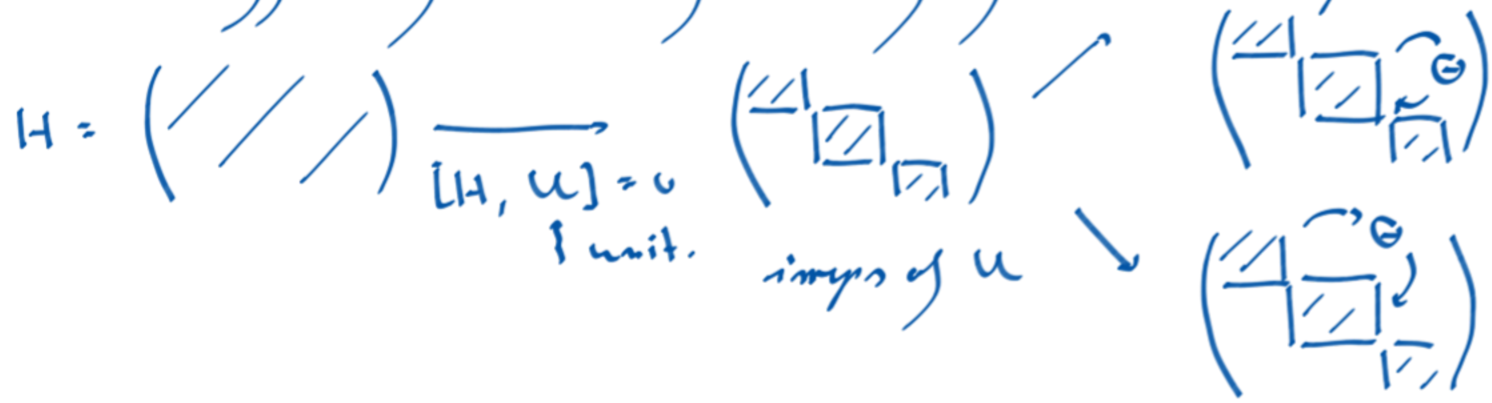
• $H = \frac{1}{2m}\left(p_x^2 + (p_y - Bx)^2\right) + V(x,y)$

• $R H R^{-1} = \frac{1}{2m}\left((-p_x)^2 + (p_y + Bx)^2\right) + V(-x,y) \neq H$

• $\Theta_T H \Theta_T^{-1} = \frac{1}{2m}\left((-p_x)^2 + (-p_y - Bx)^2\right) + V(x,y) \neq H$

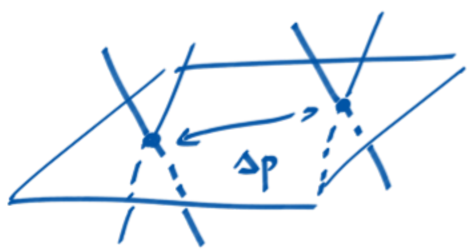
• $R\Theta_T H \Theta_T^{-1} R^{-1} = H$ ✓

• A Hamiltonian may possess anti-unitary symmetry, but the symmetry can be effectively absent if unitary symmetries are present.



Θ effectively absent.

Example:



Two Dirac nodes stabilized by translational symmetry (U = translation)

Θ_T switches between nodes.

- Symmetry operations in 2^{nd}-quantized representation:
  - Time reversal: $\mathcal{T}$: $T c_a T^{-1} = U_{T,ab}\, c_b$ with T anti-unitary, i.e. $T^{-1} z T = \bar{z}$, and unitary U_T. For spin (half) integer, $\bar{U}_T = (-1)\, U_T$

    Note: $T c_a^\dagger T^{-1} = \bar{U}_{Tab}\, c_b^\dagger$

  - Particle-hole: $\mathcal{C}$: $C c_a C^{-1} = \bar{U}_{C,ab}\, c_b^\dagger$ with C unitary, and unitary U_C. For spin (half) integer: $\bar{U}_C = (-1)\, U_C$

    Note: $C c_a^\dagger C^{-1} = U_{Cab}\, c_b$

Second vs. first quantized perspective: Warmup: consider particle number conserving Hamiltonians: $\hat{H} = c_a^\dagger H_{ab} c_b$

$$\mathcal{T}(\hat{H}) = T c_a^\dagger T^{-1} \bar{H}_{ab} T c_b T^{-1} = \bar{U}_{Tac}\, c_c^\dagger H^T_{ab}\, U_{Tbd}\, c_d$$

$$= c_c^\dagger\, U^{-1}_{Tca} H^T_{ab} U_{Tbd}\, c_d$$

$$= c_c^\dagger \left(U_T^\dagger H^T U_T\right)_{cd} c_d$$

no const

$\rightsquigarrow$ 2^{nd} quantized representation induces 1st quantized $\mathcal{T}(H) = U_T^{-1} H^T U_T$

$$\mathcal{C}(\hat{H}) = C c_a^\dagger C^{-1} H_{ab}\, C c_b C^{-1} =$$

$$= U_{Cac}\, c_c\, H_{ab}\, \bar{U}_{Cbd}\, c_d^\dagger =$$

$$= -c_d^\dagger\, U^{-1}_{Cdb} H_{ab} U_{Cac}\, c_c + \underbrace{U^{-1}_{C,cb} H_{ab} U_{C,ac}}_{\operatorname{tr} H}$$

$$= -c_d^\dagger \left(U_C^{-1} H^T U_C\right)_{dc} c_c + \operatorname{tr} H$$

Self consistent picture: $\hat{H}$ particle-hole symmetric, iff $\mathcal{C}(H) = -U_C^{-1} H^T U_C = H$

Under this condition $\operatorname{tr} H = 0$.

Some comments on first vs. second quantized perspective

- Be careful! A system with $C=+1$ need not have $C=\pm 1$ in 2nd quantized representation. Example:

$$\hat{H} = c_a^\dagger h_{ab} c_b + c_a^\dagger \Delta_{ab} c_b^\dagger + h.c. \qquad , \Delta = -\Delta^T$$

Physically: Hamiltonian of 'least' degree of symmetry (does not even conserve $\hat{N}$ = particle number.) $\mathcal{C} = \mathcal{T} = 0$ (2nd)

$$\mathcal{C}(\hat{H}) = c_a h_{ab} c_b^\dagger + c_a \Delta_{ab} c_b + h.c. \neq \hat{H}$$

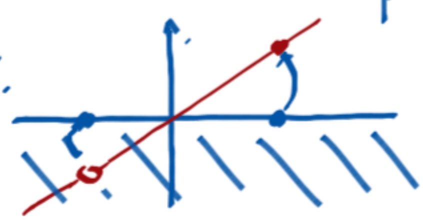
On the other hand (1st): $\hat{H} = (c^\dagger, c)_a \underbrace{\begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}_{ab}}_{H_{ab}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}_b$

$$H = -\tau_x H^T \tau_x \rightsquigarrow \mathcal{C} = +1 \checkmark$$

Nambu space

- Conclusion: A Hamiltonian (1st) may realize a symmetry which need not correspond to a physical (2nd) symmetry.

Example of a system with physical $\mathcal{C}=1$: $\hat{H} = c_p^\dagger \, p \, c_p$, i.e. linearization around some Fermi point.

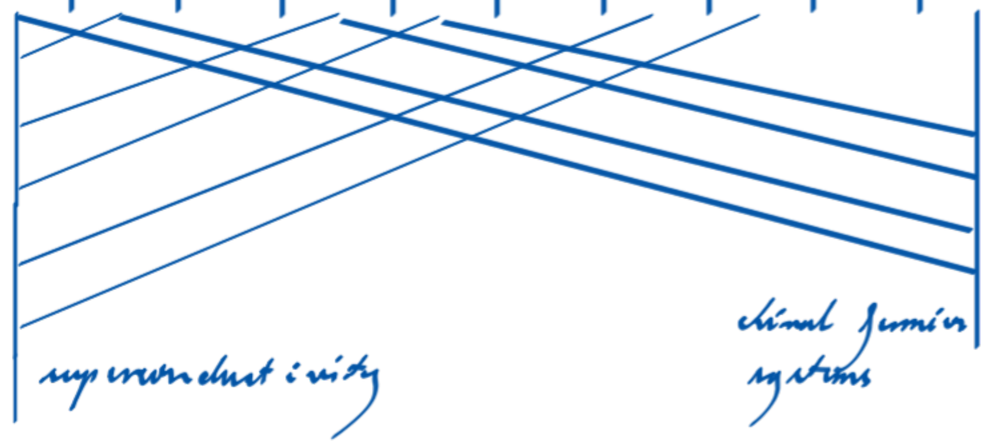


cost equal energy: $C c_p^\dagger C^{-1} = c_{-p}$

$$\mathcal{C}(\hat{H}) = \sum_p c_{-p} \, p \, c_{-p}^\dagger = -\sum_p c_{-p}^\dagger \, p \, c_{-p} + \text{const.} = + \sum_p c_p^\dagger \, p \, c_p + \text{const} = \hat{H} + \text{const.}$$

Denotation and application areas of symmetry classes

T	+1	+1	+1	-1	-1	-1	0	0	0	0
C	+1	-1	0	+1	-1	0	+1	-1	0	0
S	1	1	0	1	1	0	0	0	0	1
label	BDI	CI	AI	DIII	CII	AII	D	C	A	AIII



Properties of symmetry classes (pre-topology)

- Systems with $C = \pm 1$ (superconductors) and AIII have spectrum symmetric around 0

  $$\hat H |\chi\rangle = \epsilon |\chi\rangle \Rightarrow \hat H C|\chi\rangle = - C \hat H |\chi\rangle = - \epsilon\, C|\chi\rangle .$$

  Note: For $\epsilon = 0$ $|\chi\rangle$ and $C|\chi\rangle$ can be degenerate. This is the case with topological zero modes.

- Near symmetry point $\epsilon = 0$: quantum interference phenomena

  Consider, e.g., single particle density of states of gapless (!) superconductor

  $$\rho(\epsilon) = -\frac{1}{2\pi} \operatorname{Im} \operatorname{tr}\left(\hat G^+(\epsilon)\tau_3\right) = -\frac{1}{2\pi} \operatorname{Im} \int dx \operatorname{tr} \langle x | \hat G^+(\epsilon)\, \tau_3 | x \rangle$$

  ph

  $$\left(\hat G^+(\epsilon)\right)^{-1} = \epsilon^+ \mathbb{1} - \underbrace{\begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}}_{\hat H} \qquad \Delta = -\Delta^T$$

- Gaplessness: Δ vanishes 'on average', e.g. in d, p-wave superconductor, SN hybrid systems, ...

- Symmetry: $\tau_x \hat H^T \tau_x = - H$. With $C = \tau_x K$: $C = +1$, $T = 0$ class D write $\hat{G}^+$ as:

$$\hat{G}^+(\epsilon) = \begin{pmatrix} \underbrace{\epsilon + i0 - h}_{\text{'normal' retarded propagation of quasi-particle at energy } \epsilon} & \Delta \\ \underbrace{\Delta^+}_{\text{scattering between particle and hole}} & \underbrace{-(-\epsilon - i0 - h^T)}_{\text{propagation of hole at energy } -\epsilon \text{ backwards in time } (h^T).} \end{pmatrix}^{-1}$$

Heuristic interpretation of scattering

- Def.: $g^\pm(\epsilon) = (\epsilon^\pm - h)^{-1}$
- Consider $G^+_{12}(x,x,\epsilon)$ = amplitude for particle at energy ϵ to get scattered into hole at energy $-\epsilon$

Pictorially:

Note: $\langle x'| g^-(-\epsilon)^T |x\rangle = \overline{\langle x| g^+(\epsilon) |x'\rangle}$: hole amplitude is complex conjugate of particle amplitude

= Fourier transform of probability to propagate $x \xrightarrow{\overset{\text{time}}{t}} x'$ in time $\equiv \Pi(x,x',\epsilon)$

- The dependence of Π coordinates/energy depends on type of system (disorder, size,)

For simplicity: consider finite size system at energies $\epsilon < \Gamma(L)$ corresponding to times $t > \Gamma(L)^{-1}$: ergodic regime ~ $\Pi(x,x',t) \simeq L^{-d}\underbrace{\Theta}_{\text{Heaviside}}(t)$, independent of x,x' and normalized $\int dx'\, \Pi(x,x',t) = 1$. Fourier trafo:

$$\Pi(x,x',\epsilon) = L^{-d}\frac{1}{\epsilon^+}$$

Return amplitude singular at low energies. Observable consequences: band center anomalies in density of states and transport coefficients.

Example: $\rho(\varepsilon)$ of class D superconductor (relevant to observation of Majorana fermions c.f. 1206.0434 for the full story). $\rho(\varepsilon) = -\frac{1}{2\pi}\int dx \, \mathrm{Im}\left(\hat{G}^{+}_{11}(x,x,\varepsilon) - \hat{G}^{+}_{22}(x,x,\varepsilon)\right)$

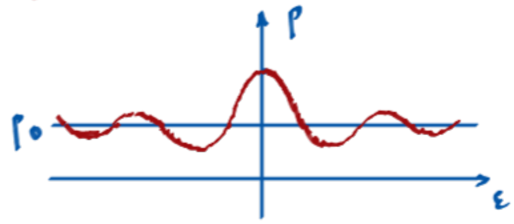
Lowest order pert. theory in Δ:

$G^{+}_{11}(x,x,\varepsilon) \sim$  $\sim \frac{1}{\varepsilon^2}$

Full resummation of series (complicated):

$$\rho(\varepsilon) = \rho_0\left(1 + \frac{\sin(\pi\varepsilon/\delta)}{\pi\varepsilon/\delta}\right)$$

$\delta = \rho_0^{-1}$ single particle level spacing



Remarks:
- Effect conceptually related to weak localization of ordinary metals
- Spectral peak looks like a zero-energy 'state'.
- Can easily be confused with topological zero mode (Majorana fermion)
- Not of topological origin.

Symmetries & Topology

The geometry of symmetry classes

Consider time evolution $\hat{U} = \exp(i\hat{H}t)$ generated by $\hat{H}$ of given symmetry. What is $\hat{U}$ geometrically? Example: • $(\hat{T}, \hat{C}, \hat{S}) = (0,0,0)$, $\hat{H} = \hat{H}^\dagger \rightsquigarrow \hat{U} \in U(n)$ (class A).

• $(\hat{T}, \hat{C}, \hat{S}) = (1,0,0)$, $\hat{H} = \hat{H}^\dagger$, $\hat{H} = \hat{H}^T \rightsquigarrow$

$$\hat{U} = \hat{U}^T \rightsquigarrow \hat{U} \in U(n)/\underbrace{O(n)}_{\text{antisymmetric matrices}} \quad (\text{class AI})$$

T	+1	+1	+1	−1	−1	−1	0	0	0	0
C	+1	−1	0	+1	−1	0	+1	−1	0	0
S	1	1	0	1	1	0	0	0	0	1
label	BDI	CI	AI	DIII	CII	AII	D	C	A	AIII
	$O(2n)/O(n)\times O(n)$	$Sp(2n)/U(n)$	$U(n)/O(n)$	$O(2n)/U(n)$	$Sp(2n)/Sp(n)\times Sp(n)$	$U(2n)/Sp(n)$	$SO(n)$	$Sp(n)$	$U(n)$	$U(2n)/U(n)\times U(n)$

- Time evolutions take values in compact symmetric spaces of rank $\sim n$.

  Symmetric spaces: • have geometry that looks 'the same' everywhere. Example: $U(2)/U(1)\times U(1) = A\mathrm{III}_2$ the 2-sphere. Heuristics:

  
  ergodic time evolution knows about symmetry, 'uniform' otherwise

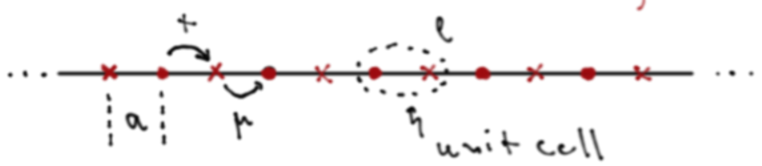
  • Have been labeled by Cartan as above

  • There are just 10 families and there are 1-1 to quantum symmetries

- Symmetric spaces: Rich structure to describe topology $\rightsquigarrow$

$\rightsquigarrow$ Two examples $\rightsquigarrow$

The Su-Schrieffer-Heeger (SSH) chain (1971)



(periodic b.c.)

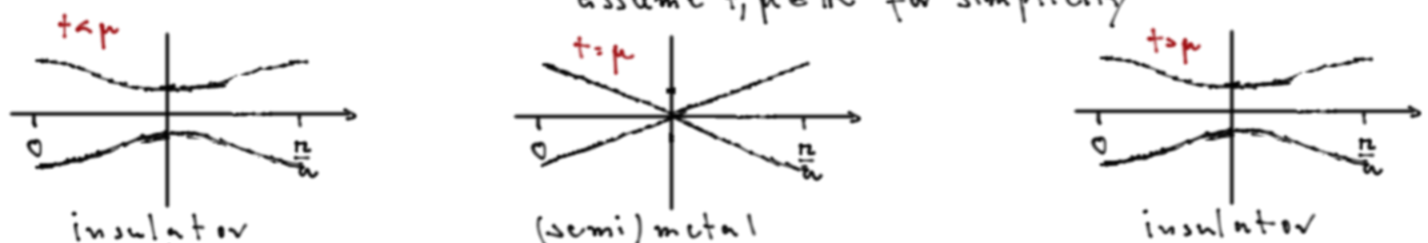
Diagonalize problem in terms of 2-component Bloch wave functions

- $\psi_k(l) = \begin{pmatrix} \psi_\times \\ \psi_\bullet \end{pmatrix}_k e^{ikl}$ $\quad k = 0, \frac{2\pi}{L}, \dots, \frac{\pi}{a}$ $\quad L = N(2a)$ = chain length

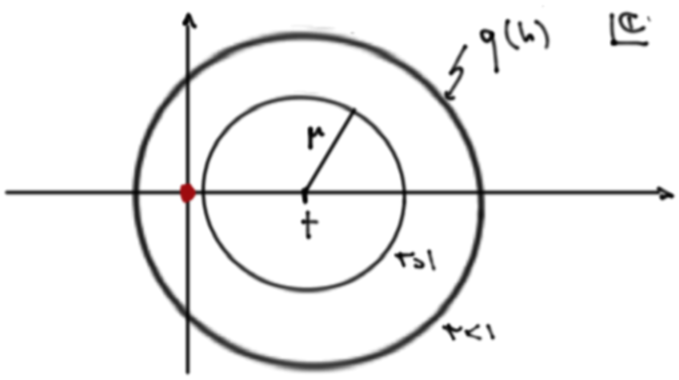
- $\hat{H}_k = \begin{array}{c} \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix} \end{array} \begin{array}{c} \times \\ \bullet \end{array}$ $\quad q_k = t - e^{ik\cdot 2a}\mu$

- Eigenvalues: $\epsilon_k^\pm = \pm |q_k| = \left(t^2 + \mu^2 - 2t\mu\cos(2ka)\right)^{1/2}$

  assume $t, \mu \in \mathbb{R}$ for simplicity



- interpretation: $r \equiv \frac{t}{\mu} = 1$ marks quantum phase transition between two distinct insulating phases. No change in symmetry, no local order parameter.
- topological order: ground states of $r<1$ and $r>1$ carry distinct topological invariant



- curve $S^1 \to \mathbb{C}\setminus\{0\}$

  $k \mapsto q(k)$

  homotopically trivial/non-trivial for $r>1$/$r<1$

- observable difference: cut system open depending on

the system with $r>1$ has/has not two zero energy boundary states sharply ($\mathcal{O}(a)$) localized at system boundaries. The insulating $r>1$ phase has a 'conducting surface'.

- Summary: $\exists$ (second order) topological quantum phase transitions without symmetry change/local order parameter.

Anomalous QH insulator

Class A in $d=2$. Q: Can there be a QH-effect without magnetic field?

A: (Haldane) Consider lattice Hamiltonian

$$\hat H = \sin k_x\, \sigma_x + \sin k_y\, \sigma_y + (r + \cos k_x + \cos k_y)\, \sigma_z \equiv v_k \cdot \sigma = U_k \sigma_z U_k^{-1} \qquad U_k \in SU(2)$$

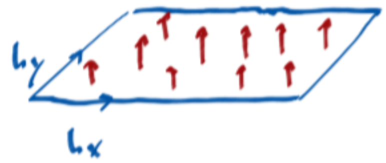
Eigenvalues: $\varepsilon_k^{\pm} = \pm\left(\sin^2 k_x + \sin^2 k_y + (r + \cos k_x + \cos k_y)^2\right)^{1/2}$ has gap around $\varepsilon = 0$ for $r \neq -2, 0, 2$. Eigenstates: $\chi_{\pm k} = U_k |\pm \tfrac{1}{2}\rangle$. Geometrically: $|\chi_{+k}\rangle =$

$|\pm\tfrac12\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

= unit vector e_z rotated into direction of v_k: a point on the Bloch sphere S^2. Ground state is map: $\chi_{-k}: T^2 \to S^2$, $k \mapsto \chi_{-k}$, $T^2 = S^1 \times S^1 = \{(k_x, k_y) \in [-\pi,\pi] \times [-\pi,\pi]\}$

For

$r \gg 2$:



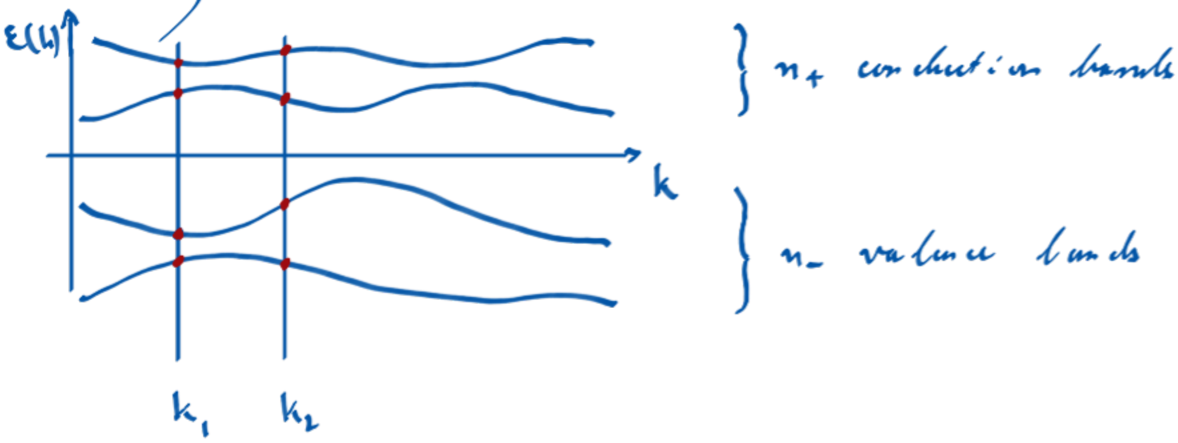
$|r| \ll 2$:

(should resemble a Hedgehog)

$\rightsquigarrow$ For $r > 2$, $r \in [0,2]$, $r \in [-2,0]$, $r < -2$, χ_{-k} has winding $0, 1, -1, 0$, resp.

Classification of topological matter I: Homotopy Theory

Schematic of insulator band structure:



Topological information encoded in evolution of ground state. For each $k \in T^d$ – d-torus of Brillouin zone, $\hat{H}_k$ diagonalized by unitary $U_k \in U(n_+ + n_-)$ (which must be compatible with symmetries). Re-ordering of occupied / unoccupied states inessential $\rightsquigarrow$ relevant information on GS encoded in elements of Grassmannian $U(n_+ + n_-)/U(n_+)\times U(n_-)$ or symmetry restricted subset thereof.

- Example: AQH: A, $n_+ = n_- = 1$ $\quad U(2)/U(1)\times U(1) = S^2$ two sphere

  SSH: AIII, $n_+ = n_- = 1$ $\quad \hat{H}_k = \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix}$ $\quad \psi_{-,k} = \begin{pmatrix} 1 \\ \underbrace{-\bar{q}_k/|q_k|}_{\in S^1} \end{pmatrix}$

  $\rightsquigarrow$ Grassmannian $\simeq S^1$

Homotopic invariants of maps $\phi: T^d \to$ Gr(assmanian) can be obtained as winding numbers or Chern numbers, or 'Chern-parities ($\mathbb{Z}_2$-insulators)'

AQH: $\quad W = \frac{1}{4\pi} \int_{T^2} dk_x\, dk_y\, \hat{v}_k \cdot (\partial_{k_x} \hat{v}_k \times \partial_{k_y} \hat{v}_k)$

SSH: $\quad W = \frac{i}{2\pi} \int_{T^1} q_k^{-1} \partial_k q_k$

Homotopy approach,

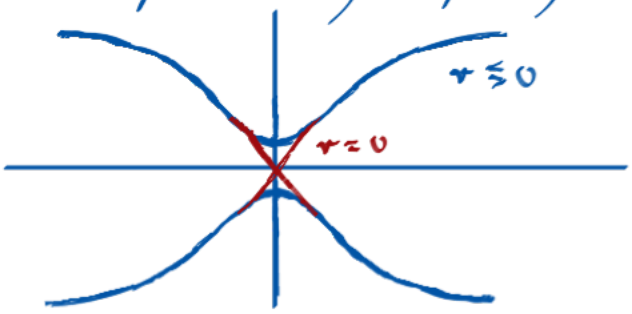
- great for classifying topologies in dependence on dimensionality / symmetry (*)
- However also abstract and
- ill suited to description of bulk boundary correspondence

(*) The systematics of maps $T^d \to Gr$ can be understood in terms of category (K-) theory. The result is the periodic table of topological insulators which describes all non-trivial homotopies (in the limit $n_\pm \to \infty$).

	d												
Cartan	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

Classification of topological matter II: Dirac-Hamiltonian approach.

Consider spectrum of topological insulator with topological phase transition point driven by parameter r.



→ appearance of 1st order 0s in Brillouin zone.

Can be described by effective Dirac-Hamiltonians ⟺ linearization of band structures around 0s.

Example SSH chain:

$$\hat{H} \underset{\substack{p=1,\ t=r \\ a=1}}{=} \begin{pmatrix} & \overset{q}{\overbrace{r-e^{ik}}} \\ r-e^{-ik} & \end{pmatrix} = (r-\cos k)\,\sigma_1 + i \sin k\,\sigma_2 \simeq$$

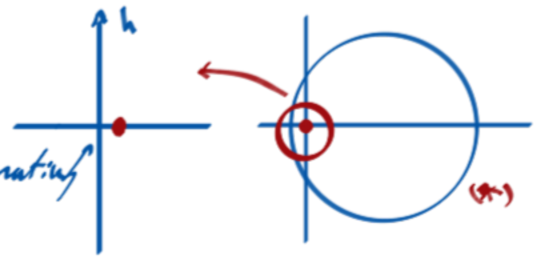
$$\underset{r=1+m}{\simeq} m\sigma_1 + ik\sigma_2$$

Topological invariant:

$$W = \frac{1}{2\pi i}\int_k q^{-1}\partial_k q = \frac{1}{4\pi i}\int_k \mathrm{tr}\left(\hat{H}^{-1}\partial_k \hat{H}\,\sigma_3\right) =$$

$$= \frac{1}{4\pi i}\int_k \mathrm{tr}\left(\left(-m\sigma_1 - ik\sigma_2\right) i\sigma_2\cdot\sigma_3\right)\frac{1}{m^2+k^2} = \frac{1}{4\pi i}\int_{-\infty}^{\infty} dk\,\frac{m}{m^2+k^2} =$$

$$= \frac{1}{2}\,\mathrm{sgn}\, m$$



linearized integration contour

- Interpretation (valid beyond SSH example)

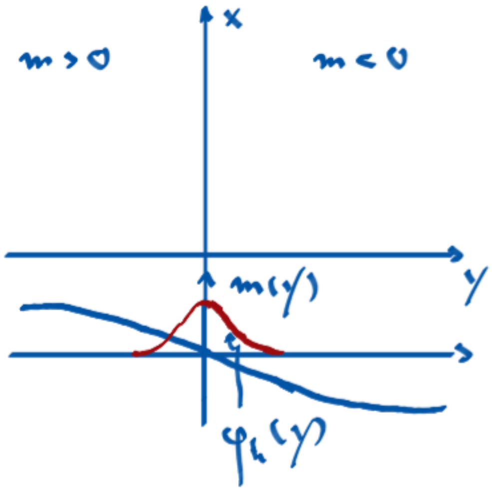
  I: Linearized integration does winding number. Misses contribution from (*). However jump $W(r>0) - W(r<0)$ is predicted correctly.

  II: Integrals can be UV-problematic. However always properly regularized by band structure ($H \to \left(m + \frac{k^2}{2}\right)\sigma_1 + ik\sigma_2$ in $O(k^2)$).

Dirac Hamiltonians and boundary physics.

Topological bulk ~ 0 energy boundary states ($d=1$), or gapless boundary modes ($d>1$)

Example $d=2$ QAH: $\hat{H} = \sin k_x \sigma_x + \sin k_y \sigma_y + (r - \cos k_x - \cos k_y)\sigma_z \underset{r=2+m}{\simeq}$



$$\simeq k_p \sigma_p + m\sigma_z$$

$$\simeq k_x \sigma_z + k_y \sigma_y + m\sigma_x$$

$\sigma_x \to \sigma_z,\ \sigma_y \to \sigma_x,\ \sigma_z \to \sigma_y$

Ansatz for wave functions: $\psi_k(x,y) = \varphi_k(y)\, e^{ik_x x}$

$$\begin{pmatrix} k_x & -i\partial_y - im \\ -i\partial_y + im & -k_x \end{pmatrix}\begin{pmatrix}\varphi_1(y)\\ \varphi_2(y)\end{pmatrix} = k_x \begin{pmatrix}\varphi_1(y)\\ \varphi_2(y)\end{pmatrix}$$

has solutions concentrated at $y=0$ boundary and linearly dispersive in x-direction (e.g. for $m(y) = -cy$, $\rightarrow (\varphi_1(y) \sim e^{-\frac{c}{2}y^2}, \varphi_2(y) = 0)$). Existence of solutions guaranteed by index theorems.

III: Dirac approach well suited to describe boundary modes

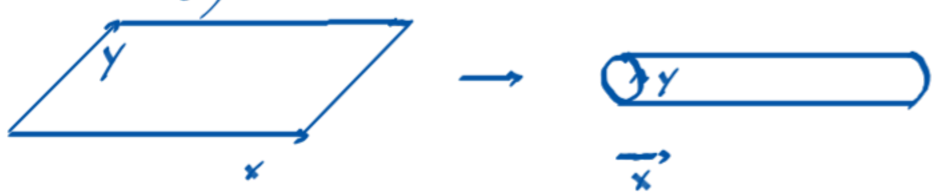
Dirac approach and periodic table.

Periodic table shows (Bott) periodic diagonal structures. E.g.:

	1	2	3	4
AIII	$\mathbb{Z}$		$\mathbb{Z}$	
A		$\mathbb{Z}$		$\mathbb{Z}$

Can be understood from Dirac approach by dimensional reduction

Example: Compactify $d=2$ QAH insulator (class A) in one direction to a cylinder



$$\hat{H}_2 = k_x \sigma_x + \sin k_y \sigma_y + (1 + m - \cos k_y)\sigma_z \xrightarrow[\text{via quantization}]{} k_x\sigma_x + m\sigma_z = \hat{H}_1 \qquad [\hat{H}_1, \sigma_y]_+ = 0$$

$\rightsquigarrow$ $\hat{H}_1$ is in AIII. Can show: non-vanishing Chern number of $\hat{H}_2$ $\rightarrow$ Non-vanishing winding number of $\hat{H}_1$

IV: Dirac insulators great for understanding dimensional structures and topology.

Classification of topological matter III: Beyond translational invariance

So far approaches emphasized momentum quantum numbers: translational invariance required.

Q: How describe topology in aperiodic structures (disorder, incommensurate potentials, irregular geometries)

Real space approaches to topology:
- scattering theory (cf. 1101.1749)
- gauge theory

topological monitor scattering matrix eigenvalues.

Gauge theory approach.

Example: Consider multi-channel class D superconductor wires

$$\hat{H}' = -\underset{\substack{\uparrow \\ \text{particle hole}}}{\tau_x} \hat{H}'^T \tau_x$$

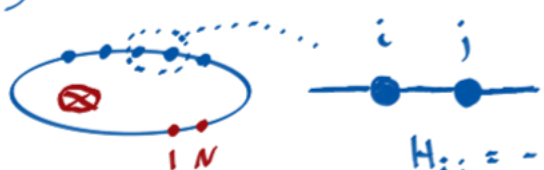
apply similarity transformation: $\hat{H} = e^{\frac{i\pi}{4}\tau_1} \hat{H}' e^{-\frac{i\pi}{4}\tau_1}$

physically: The Majorana basis: $(c, c^+) \to (\gamma \pm i\nu)\frac{1}{\sqrt{2}}$

$\hat{H}^T = -\hat{H}$ an anti-symmetric purely imaginary matrix.

Consider bilinear form: $\underset{\gamma}{\gamma^T H \gamma}$. Q.M. gauge symmetry $c \to e^{i\varphi} c$ $\varphi \in [0, 2\pi]$ broken to $\varphi \in \{0, \pi\}$. Gauge group $\mathbb{Z}_2$. Majorana fermions

Consider ring shaped quantum wire



$H_{ij} = -H_{ji}$

- Send $\mathbb{Z}_2$ gauge flux, φ
- Represent it by vector potential A: $\int_{\text{ring}} A = \varphi$.
- Deform A such that $A_{ij} = 0$ $i \neq 1, j = N$
- Flux $\varphi = \pi$ amounts to sign inversion: $H_{1N} \to -H_{1N}$
- Consider adiabatic change 

$$H_{ij} \to \overset{\pi}{\underset{\downarrow}{s}} H_{ij}$$

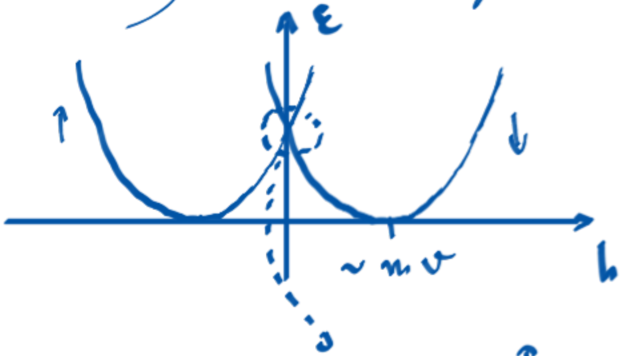
$s \in [1, 1]$ $\quad s = \overset{0}{1} \ldots, \overset{\pi}{-1}, \ldots, \overset{0 \simeq 2\pi}{1}$

- $s = 1 \to s = -1$ must cross $s = 0$, the system is cut open along the adiabatic evolution.
- Iff topological:  Adiabatic evolution must lead to emergence of 0-modes.

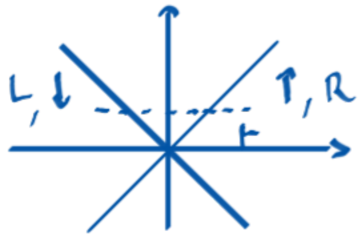
$s = 0$: emergence of zero energy Majorana states.

Majorana fermions in topological superconductor wires: bottom up

Start from 1d spin orbit coupled quantum wire



$$H = c^\dagger_{k\sigma}\left(\frac{k^2}{2m} + \underbrace{\sigma}_{\pm 1} k v\right) c_{k\sigma}$$



$$H_{eff} \overset{\vdots}{=} \sum_c \int dx\, \bar\chi_c (c\, i\partial_x + \mu)\chi_c$$

$v_F = 1$ $\qquad$ $c = L, R$

Add to the problem:

- Zeeman magnetic field in x-direction

  $\sim B\,\bar\chi\,\sigma_x\,\chi \to B\int dx\,(\bar\chi_L \chi_R + h.c.)$

- Superconductor pairing field

  $\sim (\Delta\,\bar\chi_\uparrow \bar\chi_\downarrow + h.c.) \to \int dx\,(\Delta\,\bar\chi_L \bar\chi_R + h.c.)$

Both Δ, B open gap, but these gaps fight each other $|B| = |\Delta|$: topological quantum critical point

$$\to H = \int dx\, \bar\Psi \overbrace{\begin{pmatrix} i\partial_x + \mu & & B & \Delta \\ & +i\partial_x - \mu & -\bar\Delta & -B \\ B & -\Delta & -i\partial_x + \mu & \\ \bar\Delta & -B & & -i\partial_x - \mu \end{pmatrix}}^{\hat H} \Psi$$

$$\bar\Psi = (\overbrace{\underbrace{\chi_L, \bar\chi_L}_{ph}, \bar\chi_R, \chi_R}^{LR})$$

$$H = -\sigma_x^{ph} H^T \sigma_x \qquad \text{class D}$$

Why gapless at $B = \Delta$? Switch to Majorana basis:

$$\eta_L = \frac{1}{\sqrt{2}}(\bar{\chi}_L + \chi_L) \qquad \Xi_L = \begin{pmatrix} \eta_L \\ \eta_L' \end{pmatrix}$$

$$\eta_L' = \frac{1}{\sqrt{2}i}(\bar{\chi}_L - \chi_L) \qquad \Xi = \begin{pmatrix} \Xi_L \\ \Xi_R \end{pmatrix}$$

Assume $\Delta = \epsilon\, \mathbb{R}$ for simplicity:

$$H = \int dx \; \Xi^T \, Y \, \Xi \qquad \eta\eta'$$

$$Y = \begin{pmatrix} i\partial_x + \mu\tau_2 & iB + i\Delta\tau_3 \\ -iB - i\Delta\tau_3 & -i\partial_x + \mu\tau_2 \end{pmatrix}_{L,R} \quad Y^T = -Y$$

For $\mu = 0$, $\pm B = \Delta$: massless modes appear. Assume $B = -\Delta + m$, focus on $\eta\ (\chi')$

$$H_{eff} = \int dx \; \eta^T \left(i\partial_x \underset{L,R}{\tau_3} + m\tau_2 \right) \eta$$

Edge state at topological phase transition

Assume : $m = m(x)$ $\quad m(x \to \pm\infty) = \pm m_0$

Seek for 0-energy eigenfunction of H_{eff}. Unitary transformation $\tau_3 \to \tau_1$

$$H^{1st\,q.} = \begin{pmatrix} & i\partial_x - im \\ i\partial_x + im & \end{pmatrix}$$

Def.: $\phi(x) = \int_{-L}^{x} dx'\, m(x')$ $\quad L \to \infty$ $\quad \phi$ monotonously growing

$\cdot\ \chi(x) = W e^{-\phi(x)} \quad \rightsquigarrow \quad \partial_x \chi + m\chi = 0$

$$H \begin{pmatrix} \chi \\ 0 \end{pmatrix} = 0 \quad \rightsquigarrow \quad \text{non-degenerate 0-energy Majorana state}$$